EXTENDED NEWTON METHODS FOR MULTIOBJECTIVE OPTIMIZATION: MAJORIZING FUNCTION TECHNIQUE AND CONVERGENCE ANALYSIS

JINHUA WANG*, YAOHUA HU†, CARISA KWOK WAI YU‡, CHONG LI§, AND XIAOQI YANG¶

Abstract. We consider the extended Newton method for approaching a Pareto optimum of a multiobjective optimization problem, establish quadratic convergence criteria and estimate a radius of convergence ball under the assumption that the Hessians of objective functions satisfy an L-average Lipschitz condition. These convergence theorems significantly improve the corresponding ones in [SIAM J. Optim 20 (2009), pp. 602-626]. As applications of the obtained results, convergence theorems under the classical Lipschitz condition or the γ-condition are presented for multiobjective optimization, and the global quadratic convergence results of the extended Newton method with Armijo/Goldstein/Wolfe line-search schemes are also provided.

Key words. Multiobjective optimization; Pareto optimum; Newton method; convergence criteria; L-average Lipschitz condition

AMS subject classifications. Primary, 90C29, 90C30; Secondary, 65K05

1. Introduction. Let $U \subseteq \mathbb{R}^l$ be an open set, and let $F : U \to \mathbb{R}^m$ be a twice continuously differentiable function. In the present paper, we consider the following multiobjective optimization problem:

$$\min_{x \in U} F(x). \tag{1.1}$$

This type of problems has been widely studied by [6, 8, 24] and extensively applied in various areas such as engineering [16], management science [2], environmental analysis [28], economy

*Department of Mathematics, Hangzhou Normal University, Hangzhou 311121, P. R. China (wjh@zjut.edu.cn). This author's work was supported in part by the National Natural Science Foundation of China (grant 11771397) and Zhejiang Provincial Natural Science Foundation of China (grant LY17A010021, LY17A010006).

†Corresponding author. Shenzhen Key Laboratory of Advanced Machine Learning and Applications, College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, P. R. China (mayhhu@szu.edu.cn). This author's work was supported in part by the National Natural Science Foundation of China (11601343, 11871347), Natural Science Foundation of Guangdong (2016A030310038), Natural Science Foundation of Shenzhen (JCYJ20170817100950436, JCYJ20170818091621856) and Interdisciplinary Innovation Team of Shenzhen University.

‡Department of Mathematics and Statistics, The Hang Seng University of Hong Kong, Shatin, Hong Kong (carisayu@hsu.edu.hk). This author's work was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (grant UGC/FDS14/P02/15 and UGC/FDS14/P02/17).

§School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, P. R. China (cli@zju.edu.cn). This author's work was supported in part by the National Natural Science Foundation of China (grant 11571308) and Zhejiang Provincial Natural Science Foundation of China (grant LY18A010004).

¶Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong (mayangxq@polyu.edu.hk). This author's work was supported in part by the Research Grants Council of Hong Kong (PolyU 152128/17E).

[44], radiotherapy [38], statistical regression [31] and so on. In particular, the intensity modulated radiotherapy and multiple ridge regression are reformulated by (possibly unconstrained) strongly convex multiobjective optimization problems in [38] and [31], respectively.

Motivated by its extensive applications, a great amount of attention has been attracted to the development of optimization algorithms, and many iterative methods have been proposed to approach a Pareto optimum of multiobjective optimization; see [3, 6, 7, 12, 13, 14, 15, 21, 22, 23, 41, 46] and references therein. Among them, there are mainly two different approaches for solving multiobjective optimization. One is based on the scalarization technique (see [7, 15]), the other is based on descent methods; see [3, 12, 13, 14, 22, 23]. Scalarization methods compute the Pareto or weakly Pareto solutions by choosing some parameters in advance, and reformulating them as single objective optimization problems. As shown in [22, p. 618], scalarization methods might be problematic for some examples, where most choices of the parameters lead to unbounded (and thus unsolvable) scalar problems. Usually, the descent methods do not require any parameter information. For example, a steepest descent method was proposed in [23] to solve multiobjective optimization problems, where neither ordering information (i.e., an ordering of importance of the components of the objective function vector) nor weighting factors are assumed to be known. Other descent methods such as Newton method [22], projected gradient method [12], proximal point method [4, 9], trust-region method [37] and so on, have been proposed and studied extensively for multiobjective optimization problems with an ordering defined by the non-negative orthant. Moreover, the Newton method in [22] has also been extended to solve multiobjective optimization problems with an ordering defined by a closed, convex and pointed cone or a variable ordering structure, respectively in [13, 3], and the convergence properties were studied therein.

In the present paper, we focus on the Newton method for solving multiobjective optimization problems. Its original idea is from the classical Newton method for solving nonlinear equations, the study of which has a long history; see [45]. One of the most famous results on Newton method is the well-known Kantorovich's theorem (cf. [26]), which provides a criterion ensuring the quadratic convergence under some mild conditions around the initial point x_0. Another important result is Smale's point estimate theory (i.e., α-theory and γ-theory) developed in [39, 40], which provides the rules to judge an initial point to be an approximate zero, depending on the information of the analytic nonlinear operator at this initial point or at a solution. A significant development in this direction was made by Wang in [42], where the notion of the generalized L-average Lipschitz condition was introduced for developing the convergence theory of the Newton method for solving an equation in a Banach space, and unifying the Kantorovich's theorem and the Smale's point estimate theory. Extensions of the mentioned results on the Newton method have also been made for finding the singularities of the vector fields on Riemannian manifolds [11, 20, 30].

The extended Newton method (with Armijo line-search scheme) which we considered here for solving multiobjective optimization problems was introduced by Fliege et al. [22] for unconstrained (strongly) convex multiobjective optimization problems. Compared with other iterative methods for multiobjective optimization, as pointed out in [22], the extended Newton method enjoys several advantages: (a) it has a fast convergence rate under some mild conditions; (b) its subproblems can be solved effectively; and (c) it does not require a priori weighting factor or any other priori information for the objective functions. Due to these benefits, there is a great demand for further investigating the convergence theory of the extended Newton method, which is formally stated as follows.

ALGORITHM 1.1.

Step 1. *Choose* $x_0 \in U$ *and* $\sigma \in (0,1)$*, and set* $n := 0$.
Step 2. *Solve the direction search problem*

$$\min_{s\in\mathbb{R}^l} \max_{j=1,\ldots,m} \nabla F_j(x_n)^T s + \frac{1}{2} s^T \nabla^2 F_j(x_n) s$$

to obtain its minimizer $s(x_n)$ *and its minimal value* $\theta(x_n)$.
Step 3. *If* $\theta(x_n) = 0$*, then stop; otherwise, proceed to Step 4.*
Step 4. *Choose* α_n *as the maximal value of* $\{2^{-i} : i \in \mathbb{N}\}$ *such that*

$$x_n + \alpha_n s(x_n) \in U \text{ and } F_j(x_n + \alpha_n s(x_n)) \leq F_j(x_n) + \sigma\alpha_n\theta(x_n) \text{ for all } j = 1, \ldots, m.$$

Step 5. *Define* $x_{n+1} = x_n + \alpha_n s(x_n)$ *and set* $n := n + 1$*. Go back to Step 2.*

It is worth mentioning that the idea of the extended Newton method was also proposed in [35, Section 2.5] to solve the minimax problems of continuously differentiable and convex functions.

Under the assumption that each $\nabla^2 F_j(\cdot)$ is positive definite and Lipschitz continuous on a convex subset of U (with a nonempty interior), the authors studied in [22] the convergence of Algorithm 1.1 for problem (1.1) and established three different quadratic convergence results, which are in particular as follows: the first one is a semi-local convergence theorem, in which the quadratic convergence to a local Pareto optimum is established under the assumptions, depending on a lot of parameters, at the initial point; see [22, Theorem 6.1] for details; the second one is a local convergence theorem (i.e., [22, Corollary 6.2]) that, for each local Pareto optimum x^*, there exists $r > 0$ such that the generated sequence converges to a local Pareto optimum at a quadratic rate whenever the initial point falls in $\mathbf{B}(x^*, r)$; the last one is a global convergence theorem (i.e., [22, Corollary 6.3]), in which the sequence starting from any initial point is shown to converge to a local Pareto optimum at a quadratic rate.

The purpose of the present paper is to continue the theoretical study of the extended Newton method for multiobjective optimization problems. We focus on the case when each $\nabla^2 F_j(\cdot)$ is Lipschitz continuous and develop a new approach to provide the quantitative convergence analysis for the extended Newton methods, not only for Algorithm 1.1 but also the one without the line-search scheme (see Algorithm 3.1). Under the classical Lipschitz continuity assumption for the second derivatives $\nabla^2 F_j(\cdot)$, our main results, concerning also the three types of convergence properties mentioned above, are described as follows:

- Our theorem (i.e., Theorem 4.1) regarding the semi-local convergence property provides some explicit convergence criteria, which are only based on the data at an initial point and the Lipschitz constants of the second derivatives $\nabla^2 F_j(\cdot)$ around the initial point, for ensuring the convergence (to a local Pareto optimum) of Algorithms 3.1 and 1.1.

- Our theorem (i.e., Theorem 4.2) regarding the local convergence property provides some explicit estimates, which only depend on the data of a given local Pareto optimum and the Lipschitz constants of the second derivatives $\nabla^2 F_j(\cdot)$ around the local Pareto optimum, for the radius of the convergence balls of Algorithms 3.1 and 1.1.
- Our theorem (i.e., Theorem 4.5) regarding the global convergence property provides some sufficient conditions on the cluster point for ensuring the global convergence of the extended Newton method not only with the Armijo line-search scheme (i.e., Algorithm 1.1) but also with Goldstein/Wolfe line-search schemes (i.e., Algorithm 3.2).
- The results obtained in the present paper, containing the local, semi-local and the global types, provide explicit error estimates for any sequence generated by Algorithm 3.1 or 3.2 (and so Algorithm 1.1) in terms of the corresponding parameters/modulus, which improve the corresponding ones in [22]; see Theorem 6.1 and Corollaries 6.2, 6.3 therein.

Most of results (such as Theorems 3.4, 3.5, 3.7, 3.9 and so on) in the paper are new, and some of them (i.e., Theorems 4.1, 4.2 and 4.5), where less data is required, extend/improve partially the corresponding ones in [22, Theorem 6.1 and Corollaries 6.2, 6.3] as explained in Remark 4.1; in particular, an example is provided to show the case where the convergence result in the present paper (Theorem 4.1) is available but not the one in [22, Theorem 6.1]; see Example 4.1 for details.

Another important extension of the present paper is that the L-average Lipschitz condition, introduced by Wang [42] mentioned above, is involved in the convergence analysis of the extended Newton method. This idea has been used extensively in numerical analysis and optimization problems; see [17, 18, 29, 30] and references therein, but not been found to be applied to study multiobjective optimization problems. Note that the L-average Lipschitz condition implies actually the classical Lipschitz condition (with the Lipschitz constant being the supremum of the function $L(\cdot)$ in the involved ball). However, as shown in Theorems 4.1 and 4.2, the convergence criteria and/or the radius of the convergence ball of the extended Newton method depend heavily on the choice of the function $L(\cdot)$ for the involved function F to satisfy. In fact, the larger the value of the function $L(\cdot)$, the stricter the convergence criteria and the smaller the estimated radius of the convergence ball. This means that using the classical Lipschitz condition in our theorems rather than the L-average Lipschitz condition would produce the weaker results on the convergence criteria and/or on the radius of the convergence ball. One of the main advantages of adopting the L-average Lipschitz condition is shown in Example 4.2. That is, when the theorem under the classical Lipschitz condition is not applicable, it provides the possibility to choose a suitable non-negative and monotonically increasing function L such that the convergence theorem (which we will establish under the general L-average Lipschitz condition) is applicable for ensuring the convergence of the extended Newton method.

It should be remarked that the analysis tool used in the present paper is the majorizing function technique, which deviates significantly from that of [22]. The majorizing function technique has been widely used in the convergence analysis of the Newton method for nonlinear equations [17, 19, 42, 43] and of the Gauss-Newton method for convex composite optimization [18, 29], which enables us to establish an explicit convergence criterion and provides a precise estimation of the convergence radius. To the best of our knowledge, this is the first work to develop the majorizing function technique for the convergence analysis of the extended Newton method for multiobjective optimization.

The paper is organized as follows. In Section 2, we present the notations and preliminary results to be used in the present paper. The quadratic convergence criterion and the estimation of radius of convergence ball of the extended Newton method for multiobjective optimization problems are provided in Section 3, under the L-average Lipschitz condition. In Section 4, theorems under the classical Lipschitz condition, the global quadratic convergence results of the extended Newton method and theorems under the γ-condition are presented for multiobjective optimization problems. In Section 5, a preliminary numerical study is provided to show the high efficiency of the extended Newton method for solving some convex bi-objective optimization problems.

2. Notation and preliminary results. The notations used in the present paper are standard in Euclidean spaces. As usual, for $x \in \mathbb{R}^l$ and $r > 0$, let $\mathbf{B}(x, r)$ and $\mathbf{B}[x, r]$ respectively denote the open and closed balls in $\mathbb{R}^l$, and let $\mathbb{R}^m_+$ and $\mathbb{R}^m_{++}$ denote the non-negative orthant and positive orthant of $\mathbb{R}^m$, respectively. The standard simplex in $\mathbb{R}^m$ is denoted by $\mathbf{\Delta}_m$, i.e.,

$$\mathbf{\Delta}_m := \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^{m} \lambda_i = 1\}.$$

Let $\mathbb{R}^{m\times l}$ denote the space of all $m \times l$ matrices, and let I denote the identity matrix in $\mathbb{R}^{l\times l}$. For $M \in \mathbb{R}^{m\times l}$, the range of M is denoted by $\mathbf{R}(M)$. The following lemma regarding the inverses of the perturbations of nonsingular matrix is well-known; see for example [34, p.45].

LEMMA 2.1. *Let $A, B \in \mathbb{R}^{l\times l}$ be such that A is invertible and $\|A^{-1}\|\|A - B\| < 1$. Then B is invertible and*

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|A - B\|}.$$

If A and B are additionally symmetric, then B is positive definite.

2.1. Preliminary results about multiobjective optimization. In the present paper, we consider the multiobjective optimization problem (1.1) with $U \subseteq \mathbb{R}^l$ being an open (not necessarily convex) set and $F : U \to \mathbb{R}^m$ being a vector-valued function, denoted by

$$F := (F_1, \ldots, F_m)^T, \tag{2.1}$$

where each $F_i : U \to \mathbb{R}$ is a twice continuously differentiable and real-valued function. For a convex subset $V \subseteq U$, F is said to be $\mathbb{R}^m$-convex on V if F_i is convex on V for each $i = 1, \ldots, m$. The following notions consider Pareto optima (also called efficient points).

DEFINITION 2.2. *A point $x^* \in U$ is said to be*

(a) *a (global) Pareto optimum of F on U if there does not exist $y \in U$ such that $F(x^*) - F(y) \in \mathbb{R}^m_+$ and $F(y) \ne F(x^*)$;*

(b) *a weak Pareto optimum of F on U if there does not exist $y \in U$ such that $F(x^*) - F(y) \in \mathbb{R}^m_{++}$;*

(c) *a local Pareto optimum (resp. local weak Pareto optimum) if there exists a neighborhood $V \subseteq U$ of x^* such that x^* is a Pareto optimum (resp. weak Pareto optimum) of F on V.*

Obviously, every Pareto optimum is also a weak Pareto optimum, and each local Pareto optimum is a (global) Pareto optimum if U is convex and F is $\mathbb{R}^m$-convex on U.

For each $i \in \mathbb{N} := \{1, 2, \dots\}$, $C^i(U, \mathbb{R}^m)$ denotes the set of i-th continuously differentiable functions from U to $\mathbb{R}^m$. Let $x \in U$, $f \in C^2(U, \mathbb{R})$ and $F \in C^2(U, \mathbb{R}^m)$ given by (2.1). We use $\nabla f(x) \in \mathbb{R}^l$ and $\nabla^2 f(x) \in \mathbb{R}^{l\times l}$ to denote the gradient and the Hessian of f at x, respectively; while, the Jacobian and the second derivative of F at x are denoted by $\mathrm{D}F(x)$ and $\mathrm{D}^2F(x)$, respectively, that is,

$$\mathrm{D}F(x) = (\nabla F_1(x), \dots, \nabla F_m(x))^T \quad \text{and} \quad \mathrm{D}^2F(x) = (\nabla^2 F_1(x), \dots, \nabla^2 F_m(x))^T.$$

We say that $\mathrm{D}^2F(x)$ is positive definite if so is each $\nabla^2 F_i(x)$.

The notion of a critical point is recalled in the following definition, which characterizes a necessary (but in general not sufficient) condition for Pareto optimality and was used in [23] and [22] to investigate a steepest descent algorithm and an extended Newton method for multiobjective optimization, respectively.

DEFINITION 2.3. *A point $\bar{x} \in U$ is said to be a critical point of F if $\mathbf{R}(\mathrm{D}F(\bar{x})) \cap (-\mathbb{R}^m_{++}) = \emptyset$.*

Note that, in the case when $m = 1$, $\mathbf{R}(\mathrm{D}F(\bar{x})) \cap (-\mathbb{R}^m_{++}) = \emptyset$ is reduced to the classical optimality condition of scalar optimization. It follows from [22, Theorem 3.1] that, if $F \in C^2(U, \mathbb{R}^m)$ and $x^* \in U$ is such that $\mathrm{D}^2F(x^*)$ is positive definite, then

$$x^* \text{ is a critical point of } F \quad \Leftrightarrow \quad x^* \text{ is a local Pareto optimum of } F. \tag{2.2}$$

Following [22], associated to (1.1), we consider, for a point $x \in U$ such that $\mathrm{D}^2F(x)$ is positive definite, the following optimization problem:

$$\min_{s\in\mathbb{R}^l} \max_{j=1,\dots,m} \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s, \tag{2.3}$$

the solution of which is the Newton direction of the extended Newton method. By the positive definiteness of the Hessians, the function $s \mapsto \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s$ is strongly convex for each $j = 1, \dots, m$, and so, problem (2.3) has a unique minimizer. Let $V \subseteq U$ be convex such that $\mathrm{D}^2F(x)$ is positive definite for each $x \in V$. We use the functions $s : V \to \mathbb{R}^l$ and $\theta : V \to \mathbb{R}$ to denote the unique minimizer and the minimal value of problem (2.3), respectively, that is, for each $x \in V$,

$$s(x) := \arg\min_{s\in\mathbb{R}^l} \max_{j=1,\dots,m} \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s, \tag{2.4}$$

$$\theta(x) := \min_{s\in\mathbb{R}^l} \max_{j=1,\dots,m} \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s. \tag{2.5}$$

By the KKT optimality condition for problem (2.3), for each $x \in V$, there exist parameters $\lambda(:= \lambda(x)) \in \mathbf{\Delta}_m$ such that (see [22] for details)

$$s(x) = -\left[\sum_{j=1}^{m} \lambda_j(x) \nabla^2 F_j(x)\right]^{-1} \sum_{j=1}^{m} \lambda_j(x) \nabla F_j(x). \tag{2.6}$$

We end this subsection by recalling in the following lemmas some useful properties of the functions $s(x)$ and $\theta(x)$. Lemma 2.4 is taken from [22, Lemma 3.2].

LEMMA 2.4. *Let $V \subseteq U$ be convex and let $\bar{x} \in V$. Suppose that $\mathrm{D}^2 F(\bar{x})$ is positive definite. Then the following statements are true.*

(i) $\theta(\bar{x}) \leq 0$.
(ii) *$\bar{x}$ is not a critical point* $\Leftrightarrow$ $[\theta(\bar{x}) < 0]$ $\Leftrightarrow$ $[s(\bar{x}) \neq 0]$.
(iii) *If $\mathrm{D}^2 F(x)$ is positive definite for each $x \in V$, then s is bounded on any compact subset of V and θ is continuous on V.*

Let $F := (F_1, \ldots, F_m)^T \in C^2(U, \mathbb{R}^m)$. Throughout the whole paper, we define

$$F_\lambda(\cdot) := \sum_{j=1}^{m} \lambda_j F_j(\cdot) \quad \text{for each } \lambda := (\lambda_1, \ldots, \lambda_m)^T \in \mathbf{\Delta}_m. \tag{2.7}$$

Let $\lambda \in \mathbf{\Delta}_m$ and $x \in U$, and let $\rho_{\min}(\lambda, x)$ and $\rho_{\max}(\lambda, x)$ denote the minimum and maximum eigenvalues of the matrix $\nabla^2 F_\lambda(x)$, respectively, that is,

$$\rho_{\min}(\lambda, x) := \min\{z^T \nabla^2 F_\lambda(x) z : \|z\| = 1\} = \|\nabla^2 F_\lambda(x)^{-1}\|^{-1}$$

and

$$\rho_{\max}(\lambda, x) := \max\{z^T \nabla^2 F_\lambda(x) z : \|z\| = 1\} = \|\nabla^2 F_\lambda(x)\|. \tag{2.8}$$

Relation (2.9) and the first inequality of (2.10) in the following lemma are known in [22, Lemmas 4.2 and 4.3]; while the second inequality of (2.10) is a direct consequence of the first inequality of (2.9) and the first inequality of (2.10).

LEMMA 2.5. *Let $x \in U$ and let $\lambda \in \mathbf{\Delta}_m$ be such that $\nabla^2 F_\lambda(x)$ is positive definite. Then the following relations hold:*

$$\frac{\rho_{\min}(\lambda, x)}{2} \|s(x)\|^2 \leq |\theta(x)| \leq \frac{\rho_{\max}(\lambda, x)}{2} \|s(x)\|^2, \tag{2.9}$$

$$|\theta(x)| \leq \frac{1}{2} \|\nabla^2 F_\lambda(x)^{-1}\| \|\nabla F_\lambda(x)\|^2 \quad \textit{and} \quad \|s(x)\| \leq \|\nabla^2 F_\lambda(x)^{-1}\| \|\nabla F_\lambda(x)\|. \tag{2.10}$$

2.2. Preliminary results about majorizing function. To study the convergence properties of the extended Newton method for multiobjective optimization, we first recall some auxiliary results of a majorizing function. The majorizing function, originally introduced by Wang

[42], is a powerful tool for the study of convergence criteria of the Newton method. Let $R > 0$ and let $L : [0, R) \to \mathbb{R}_+$ be a nondecreasing and integrable function. Let $a > 0$ satisfy

$$\frac{1}{R}\int_0^R L(u)(R-u)\mathrm{d}u > \frac{1}{a}. \tag{2.11}$$

Associated to the triple $(a, \beta; L)$, we define the pair of positive constants (r_a, b_a) and the majorizing function $h_a : [0, R) \to \mathbb{R}$ by

$$a\int_0^{r_a} L(u)\mathrm{d}u = 1, \qquad b_a = a\int_0^{r_a} L(u)u\mathrm{d}u \tag{2.12}$$

(noting that r_a, b_a are well defined by (2.11) and $b_a < r_a$; see [42, Lemma 1.2] and [29, p. 615]), and

$$h_a(t) := \beta - t + a\int_0^t L(u)(t-u)\mathrm{d}u \quad \text{for each } t \in [0, R), \tag{2.13}$$

respectively. Then, we have $\int_0^{r_a} L(u)\mathrm{d}u < \int_0^R L(u)\mathrm{d}u$ and so $r_a < R$ (as L is positive) because $\int_0^{r_a} L(u)\mathrm{d}u = \frac{1}{a} < \frac{1}{R}\int_0^R L(u)(R-u)\mathrm{d}u$ by (2.12) and (2.11). Note that h_a is twice differentiable on $[0, R)$ with its derivatives being given by

$$h_a'(t) = a\int_0^t L(u)\mathrm{d}u - 1 \quad \text{and} \quad h_a''(t) = aL(t) \quad \text{for each } t \in [0, R), \tag{2.14}$$

where and throughout the whole paper, $h_a'(0)$ means the right derivative of h_a at 0.

Let $\{t_{a,n}\}$ denote a sequence generated by the classical Newton method for approaching the zeros of the majorizing function h_a with the initial value $t_{a,0} = 0$. That is,

$$t_{a,n+1} := t_{a,n} - h_a'(t_{a,n})^{-1}h_a(t_{a,n}) \quad \text{for each } n \in \mathbb{N}. \tag{2.15}$$

Some properties of the majorizing function h_a and the sequence $\{t_{a,n}\}$ are presented in the following proposition, which will be useful in the quantitative convergence analysis of extended Newton method. Part (i) of Proposition 2.6 is taken from [42, Lemma 1.2], while part (ii) is well-known in the literature of the Newton method (cf. [42]).

Proposition 2.6. *Suppose that $0 \le \beta \le b_a$. Then, the following assertions are true.*

(i) *h_a is strictly decreasing on $[0, r_a]$ and strictly increasing on $[r_a, R)$ with*

$$h_a(\beta) > 0, \quad h_a(r_a) = \beta - b_a \le 0 \quad \textit{and} \quad \lim_{t \to R^-} h_a(t) > \beta > 0.$$

Moreover, if $\beta < b_a$, then h_a has two zeros r_a^* and r_a^{**} such that

$$\beta < r_a^* < \frac{r_a}{b_a}\beta < r_a < r_a^{**}; \tag{2.16}$$

if $\beta = b_a$, then h_a has a unique zero $r_a^* \in (\beta, R)$ (in fact, $r_a^* = r_a$).

(ii) *$\{t_{a,n}\}$ is monotonically increasing and converges to r_a^*.*

(iii) *If* $\beta < b_\alpha$, *then*

$$\lim_{n\to\infty}\frac{2t_{a,n+1}-t_{a,n}-r_a^*}{t_{a,n+1}-t_{a,n}}=1 \quad and \quad \overline{\lim_{n\to\infty}}\frac{r_a^*-t_{a,n+1}}{(2t_{a,n+1}-t_{a,n}-r_a^*)^2}\le -\frac{aL(r_a^*)}{2h'(r_a^*)}. \tag{2.17}$$

Proof. To complete the proof, we only need to show assertion (iii). For simplicity, we omit the first subscript a in the sequence $\{t_{a,n}\}$, namely, write $\{t_n\}$ for $\{t_{a,n}\}$. Then, one has by (2.15) and assertion (ii) of this proposition that

$$\lim_{n\to\infty}\frac{2t_{n+1}-t_n-r_a^*}{t_{n+1}-t_n}=2+\lim_{n\to\infty}\frac{1}{-h_a'(t_n)^{-1}\frac{h_a(t_n)-h_a(r_a^*)}{t_n-r_a^*}}=1,$$

that is, the equality of (2.17) holds. On the other hand, note again by (2.15) that

$$\begin{aligned}
r_a^*-t_{n+1} &= r_a^*-t_n+h_a'(t_n)^{-1}h_a(t_n)\\
&= -h_a'(t_n)^{-1}\int_0^1[h_a'(t_n+t(r_a^*-t_n))-h_a'(t_n)](r_a^*-t_n)\mathrm{d}t\\
&= -h_a'(t_n)^{-1}\int_0^1\int_0^1 h_a''(t_n+\tau t(r_a^*-t_n))t(r_a^*-t_n)\mathrm{d}\tau(r_a^*-t_n)\mathrm{d}t\\
&\le -h_a'(t_n)^{-1}\frac{aL(r_a^*)}{2}(r_a^*-t_n)^2,
\end{aligned}$$

where the inequality holds because $h_a'(t_n)<0$ (cf. (2.12) and (2.14)), $h_a''(\cdot)=aL(\cdot)$ (cf. (2.14)) and $L(\cdot)$ is nondecreasing. Then, we obtain

$$\frac{r_a^*-t_{n+1}}{(2t_{n+1}-t_n-r_a^*)^2}\le\frac{-h_a'(t_n)^{-1}\frac{aL(r_a^*)}{2}(r_a^*-t_n)^2}{(-2h_a'(t_n)^{-1}h_a(t_n)+t_n-r_a^*)^2}=\frac{-h_a'(t_n)^{-1}\frac{aL(r_a^*)}{2}}{(-2h_a'(t_n)^{-1}\frac{h_a(t_n)-h_a(r_a^*)}{t_n-r_a^*}+1)^2},$$

and thus, the inequality of (2.17) is seen to hold. The proof is complete. □

The following lemma is useful for the convergence analysis of Newton method and is taken from [42, pp. 175]. Recall that $R>0$ and $L:[0,R)\to\mathbb{R}_+$ is a nondecreasing and integrable function.

LEMMA 2.7. *Let* $0\le\zeta<R$, *and let* $\varphi:(0,R-\zeta)\to\mathbb{R}_+$ *be defined by*

$$\varphi(t):=\frac{1}{t^2}\int_0^t L(\zeta+u)(t-u)\mathrm{d}u \quad \text{for each } 0<t<R-\zeta.$$

Then, φ *is increasing on* $(0,R-\zeta)$.

3. Convergence analysis of the extended Newton method. This section aims to establish the quadratic convergence criterion of the extended Newton method (without or with line-search scheme) for multiobjective optimization under an L-average Lipschitz condition. The extended Newton method without line-search scheme for solving the multiobjective optimization problem (1.1) is formally stated as follows.

ALGORITHM 3.1.

Step 1. *Choose* $x_0\in U$ *and set* $n:=0$.
Step 2. *Solve problem* (2.3) *at* x_n *to obtain* $s(x_n)$ *as in* (2.4).
Step 3. *Update* $x_{n+1}:=x_n+s(x_n)$ *and set* $n:=n+1$. *Go back to Step 2.*

The Armijo rule, the Goldstein rule and the Wolfe rule are three popular and typical line-search rules for the descent method for solving scalar optimization problems; see [1, 27, 33]. Below, we extend these three line-search schemes for the extended Newton method for solving multiobjective optimization problems.

DEFINITION 3.1. *Let $\sigma \in (0,1)$ and let $\nu \in (\sigma,1)$. Given $n \in \mathbb{N}$ and $x_n \in U$. Let $s(x_n)$ and $\theta(x_n)$ be given by* (2.4) *and* (2.5), *respectively. A stepsize $\alpha_n \in (0,+\infty)$ such that $x_n + \alpha_n s(x_n) \in U$ is said to satisfy*

(i) *the Armijo rule if*

$$\alpha_n = \max\{2^{-i} : i \in \mathbb{N},\ F_j(x_n + 2^{-i}s(x_n)) \le F_j(x_n) + \sigma 2^{-i}\theta(x_n) \quad \text{for all } j = 1,\dots,m\};$$

(ii) *the Goldstein rule if α_n satisfies*

$$F_j(x_n + \alpha_n s(x_n)) \le F_j(x_n) + \sigma\alpha_n\theta(x_n) \quad \text{for all } j = 1,\dots,m \tag{3.1}$$

and

$$F_j(x_n + \alpha_n s(x_n)) \ge F_j(x_n) + \nu\alpha_n\theta(x_n) \quad \text{for all } j = 1,\dots,m;$$

(iii) *the Wolfe rule if α_n satisfies* (3.1) *and*

$$\nabla F_j(x_n + \alpha_n s(x_n))^T s(x_n) \ge \nu\theta(x_n) \quad \text{for all } j = 1,\dots,m.$$

The extended Newton method with line-search scheme for solving the multiobjective optimization problem (1.1) is formally stated as follows.

ALGORITHM 3.2.

Step 1. *Choose $x_0 \in U$, $\sigma \in (0,1)$, $\nu \in (\sigma,1)$ and set $n := 0$.*
Step 2. *Solve problem* (2.3) *at x_n to obtain $s(x_n)$ and $\theta(x_n)$ as in* (2.4) *and* (2.5), *respectively.*
Step 3. *If $\theta(x_n) = 0$, then stop. Otherwise, proceed to Step 4.*
Step 4. *If $x_n + s(x_n) \in U$ and*

$$F_j(x_n + s(x_n)) \le F_j(x_n) + \sigma\theta(x_n) \quad \text{for all } j = 1,\dots,m,$$

then set $x_{n+1} := x_n + s(x_n)$, and go to Step 6. Otherwise, go to Step 5.
Step 5. *Choose a stepsize $\alpha_n \in (0,+\infty)$ satisfying the Armijo rule, or the Goldstein rule, or the Wolfe rule. Set $x_{n+1} := x_n + \alpha_n s(x_n)$.*
Step 6. *Set $n := n + 1$. Go back to Step 2.*

Obviously, a sequence generated by Algorithm 1.1 can be regarded as the one generated by Algorithm 3.2 with Step 5 using the Armijo rule.

REMARK 3.1. The major computation cost of Algorithms 3.1 and 3.2 is on solving the subproblem (2.3) at each iteration. Since it is a minimax problem of convex quadratic functions, there are many effective algorithms for solving problem (2.3) (see, e.g., [33, 35]), and thus, the resulting extended Newton method is practically attractive in applications. In particular, problem (2.3) can be reformulated as

$$\begin{array}{ll}\min & \rho \\ \text{s.t.} & \nabla F_j(x)^T s + \frac{1}{2} s^T \nabla^2 F_j(x) s - \rho \leq 0, \quad j = 1, \ldots, m, \\ & (\rho, s) \in \mathbb{R} \times \mathbb{R}^n,\end{array} \tag{3.2}$$

which is a standard convex quadratically constrained quadratic problem (QCQP). The QCQP can be cast into the semidefinite programming (SDP) and thus can be solved efficiently by several classical algorithms such as the interior point method and the path following method; see, e.g., [5, 32]. Hence, solving problem (2.3) can be implemented via several popular Matlab-based solvers such as CVX*, MOSEK†, TOMLAB‡. The numerical experiments in Section 5 validate the high efficiency of applying CVX in solving problem (2.3) for some examples.

The notion of the L-average Lipschitz condition was introduced by Wang in [42] (but using the terminology "the center Lipschitz condition in the inscribed sphere with L-average") and has been widely used to analyze the convergence properties of the Newton method; see [29, 30] and references therein. We extend in the following definition the notion of the L-average Lipschitz condition to the setting of vector-valued functions. Recall that $F := (F_1, \ldots, F_m)^T \in C^2(U, \mathbb{R}^m)$, and that $L : [0, R) \to \mathbb{R}_+$ is nondecreasing and integrable.

DEFINITION 3.2. *Let $x_0 \in U$ and $r \in (0, R)$ be such that $\mathbf{B}(x_0, r) \subseteq U$. $\mathrm{D}^2 F$ is said to satisfy the L-average Lipschitz condition on $\mathbf{B}(x_0, r)$ if, for each $i = 1, \ldots, m$ and any $x, y \in \mathbf{B}(x_0, r)$ with $\|x - x_0\| + \|y - x\| < r$, the following inequality holds:*

$$\|\nabla^2 F_i(y) - \nabla^2 F_i(x)\| \leq \int_{\|x - x_0\|}^{\|x - x_0\| + \|y - x\|} L(u) \mathrm{d}u.$$

By definition, we can check that the L-average Lipschitz condition on $\mathbf{B}(x_0, r)$ implies the classical Lipschitz condition with Lipschitz constant being $L(r)$. The introduction of the L-average Lipschitz condition is beneficial to provide the more precise convergence criterion and estimation of the convergence radius for the Newton method.

Fixing the triple $(x; a, r)$ with $x \in U$ and $(a, r) \in \mathbb{R}_+^2$, we consider the following assumption for $F \in C^2(U, \mathbb{R}^m)$ associated to the triple $(x; a, r)$ and L:

- $L : [0, R) \to \mathbb{R}_+$ is nondecreasing and integrable;
- a satifies (2.11), and $\mathrm{D}^2 F(x)$ is positive definite with each $\|\nabla^2 F_i(x)^{-1}\| \leq a$; (3.3)
- $\mathrm{D}^2 F(\cdot)$ satisfies the L-average Lipschitz condition on $\mathbf{B}(x, r) \subseteq U$.

LEMMA 3.3. *Suppose that F satisfies assumption* (3.3) *associated to $(x_0; a, r)$ and L, and that $r \leq r_a$. Let $x \in \mathbf{B}(x_0, r)$, $\lambda \in \mathbf{\Delta}_m$ and F_λ be defined by* (2.7)*. Then $\nabla^2 F_\lambda(x)$ is positive*

*http://cvxr.com/cvx/
†https://www.mosek.com/
‡https://tomopt.com/tomlab/

definite, and

$$\|\nabla^2 F_\lambda(x)^{-1}\| \le \frac{\|\nabla^2 F_\lambda(x_0)^{-1}\|}{1-a\int_0^{\|x_0-x\|} L(u)\mathrm{d}u} \le \frac{a}{1-a\int_0^{\|x_0-x\|} L(u)\mathrm{d}u}.$$

Proof. By assumption, one has that

$$\|\nabla^2 F_\lambda(x_0)^{-1}\|\|\nabla^2 F_\lambda(x)-\nabla^2 F_\lambda(x_0)\| \le a\int_0^{\|x_0-x\|} L(u)\mathrm{d}u < a\int_0^{r_a} L(u)\mathrm{d}u = 1$$

(by (2.12)). Hence, Lemma 2.1 is applicable and the conclusions hold. □

3.1. Convergence criterion. One of the main results of this subsection is presented in the following theorem, in which we provide a quadratic convergence criterion of the extended Newton method for multiobjective optimization under the assumption that the Hessians of objective functions satisfy the L-average Lipschitz condition. Theorem 3.4 not only extends [22, Theorem 6.1] under a weaker condition, but also improves it in the sense that the quantitative convergence result is provided here (see (3.7) below).

THEOREM 3.4. *Suppose that*

$$\|s(x_0)\| \le \beta \le b_a \tag{3.4}$$

and F satisfies assumption (3.3) associated to $(x_0; a, r_a^*)$ and L. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 with initial point x_0 is well-defined, stays in $\mathbf{B}(x_0, r_a^*)$, and converges to a local Pareto optimum $\bar{x} \in \mathbf{B}[x_0, r_a^*]$. Moreover, the following error estimates hold for each $n \ge 0$:

$$\|x_{n+1}-x_n\| = \|s(x_n)\| \le t_{a,n+1} - t_{a,n}, \tag{3.5}$$

and

$$\|x_n - \bar{x}\| \le r_a^* - t_{a,n}. \tag{3.6}$$

Moreover, if $\beta < b_a$, then there exists $N \in \mathbb{N}$ such that

$$\|x_{n+1}-\bar{x}\| \le \frac{r_a^* - t_{a,n+1}}{(2t_{a,n+1} - t_{a,n} - r_a^*)^2}\|x_n - \bar{x}\|^2 \quad \text{for each } n \ge N, \tag{3.7}$$

and so $\{x_n\}$ converges quadratically to $\bar{x}$.

Proof. Since $r_a^* \le r_a$ (cf. Proposition 2.6(i)), Lemma 3.3 is applicable to concluding that

$$\nabla^2 F_\lambda(x) \text{ is positive definite for any } x \in \mathbf{B}(x_0, r_a^*) \text{ and } \lambda \in \mathbf{\Delta}_m. \tag{3.8}$$

Furthermore, by assumption (3.3), it is easy to see that there exists a constant $c > 0$ such that

$$\sup_{\lambda \in \mathbf{\Delta}_m, x \in \mathbf{B}(x_0, r_a^*)} \rho_{\max}(\lambda, x) \le c, \tag{3.9}$$

where $\rho_{\max}(\lambda, x)$ is given by (2.8). We first show that $\{x_n\}$ is well-defined and (3.5). For simplicity, we, as before, omit the first subscript a in the sequence $\{t_{a,k}\}$, write $\{t_k\}$ for $\{t_{a,k}\}$. Thus, in view of Algorithm 3.1, (3.8) and (3.4), one has that x_1 is well-defined and $\|x_1 - x_0\| = \|s(x_0)\| \le \beta = t_1 - t_0$ (due to (2.15)), namely (3.5) holds for $n = 0$. Fix $k \in \mathbb{N}$. Below, we show the following implication:

$$\begin{aligned}&[x_n \text{ is well-defined for all } n = 0, 1, \ldots, k+1 \text{ and (3.5) holds for all } n = 0, \ldots, k]\\ &\Rightarrow\ x_{k+2} \text{ is well-defined and } \|s(x_{k+1})\| \le (t_{k+2} - t_{k+1}) \left(\tfrac{\|s(x_k)\|}{t_{k+1} - t_k}\right)^2.\end{aligned} \tag{3.10}$$

Granting this, $\{x_n\}$ is well-defined and (3.5) is shown by mathematical induction. To proceed, suppose that x_n is well-defined for all $n = 0, 1, \ldots, k+1$ and (3.5) holds for all $n = 0, \ldots, k$. Recall from (2.6) that there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbf{\Delta}_m$ such that

$$s(x_k) = -\left[\sum_{j=1}^{m} \lambda_j \nabla^2 F_j(x_k)\right]^{-1} \sum_{j=1}^{m} \lambda_j \nabla F_j(x_k) = -\nabla^2 F_\lambda(x_k)^{-1} \nabla F_\lambda(x_k). \tag{3.11}$$

Note by the induction assumption that

$$\|x_{k+1} - x_0\| \le \sum_{i=0}^{k} \|x_{i+1} - x_i\| \le \sum_{i=0}^{k} (t_{i+1} - t_i) = t_{k+1} < r_a^* \tag{3.12}$$

(by Proposition 2.6(ii)). Consequently, $x_{k+1} \in \mathbf{B}(x_0, r_a^*)$. Thus, in view of Algorithm 3.1, (3.8) and (3.4), one has that x_{k+2} is well-defined. Furthermore, Lemma 3.3 is applicable to concluding that

$$\|\nabla^2 F_\lambda(x_{k+1})^{-1}\| \le \frac{a}{1 - a \int_0^{\|x_{k+1} - x_0\|} L(u) \mathrm{d}u} \le -a\, {h_a}'(t_{k+1})^{-1}, \tag{3.13}$$

because, by (2.14),

$$-{h_a}'(t_{k+1})^{-1} = \frac{1}{1 - a \int_0^{t_{k+1}} L(u) \mathrm{d}u}.$$

Observe further from (3.11) that

$$\nabla^2 F_\lambda(x_k) s(x_k) + \nabla F_\lambda(x_k) = 0.$$

Thus, by the L-average Lipschitz condition assumption, we obtain

$$\begin{aligned}\|\nabla F_\lambda(x_{k+1})\| &= \|\nabla F_\lambda(x_k + s(x_k)) - (\nabla^2 F_\lambda(x_k) s(x_k) + \nabla F_\lambda(x_k))\|\\ &\le \textstyle\int_0^1 \|\nabla^2 F_\lambda(x_k + t s(x_k)) - \nabla^2 F_\lambda(x_k)\| \|s(x_k)\| \mathrm{d}t\\ &\le \textstyle\int_0^1 \int_{\|x_k - x_0\|}^{\|x_k - x_0\| + t\|s(x_k)\|} L(u) \mathrm{d}u \|s(x_k)\| \mathrm{d}t\\ &= \textstyle\int_0^{\|s(x_k)\|} L(\|x_k - x_0\| + u)(\|s(x_k)\| - u) \mathrm{d}u.\end{aligned} \tag{3.14}$$

Since by inductive assumption that $\|s(x_k)\| \le t_{k+1} - t_k$, it follows from Lemma 2.7 and (3.12) (with k in place of $k+1$) that

$$\int_0^{\|s(x_k)\|} L(\|x_k - x_0\| + u)(\|s(x_k)\| - u) \mathrm{d}u \le \frac{\|s(x_k)\|^2}{(t_{k+1} - t_k)^2} \int_0^{t_{k+1} - t_k} L(t_k + u)(t_{k+1} - t_k - u) \mathrm{d}u.$$

Note by (2.13)-(2.14) that

$$a\int_0^{t_{k+1}-t_k} L(t_k+u)(t_{k+1}-t_k-u)\mathrm{d}u = h_a(t_{k+1})-h_a(t_k)-{h_a}'(t_k)(t_{k+1}-t_k) = h_a(t_{k+1}), \quad (3.15)$$

where the last equality holds because $t_{k+1}-t_k = -h_a'(t_k)^{-1}h_a(t_k)$ (see (2.15)). Hence, we have from (3.14)-(3.15) that

$$a\|\nabla F_\lambda(x_{k+1})\| \le \frac{\|s(x_k)\|^2}{(t_{k+1}-t_k)^2}h_a(t_{k+1}).$$

Note by (2.10) that $\|s(x_{k+1})\| \le \|\nabla^2 F_\lambda(x_{k+1})^{-1}\|\|\nabla F_\lambda(x_{k+1})\|$. It follows from (3.13) that

$$\|s(x_{k+1})\| \le -{h_a}'(t_{k+1})^{-1}h_a(t_{k+1})\frac{\|s(x_k)\|^2}{(t_{k+1}-t_k)^2} = (t_{k+2}-t_{k+1})\left(\frac{\|s(x_k)\|}{t_{k+1}-t_k}\right)^2.$$

Thus, implication (3.10) is proved.

Now, we show the convergence of $\{x_n\}$ to a local Pareto optimum. Since $\{t_n\}$ is monotonically increasing and converges to r_a^* (by Proposition 2.6(ii)), (3.5) shows that $\{x_n\}$ is a Cauchy sequence, and so, there exists $\bar{x} \in \mathbf{B}[x_0, r_a^*]$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Furthermore, (3.5) says that $\lim_{n\to\infty}\|s(x_n)\| = 0$. Observe further from (2.9) and (3.9) that $|\theta(x_n)| \le \frac{c}{2}\|s(x_n)\|^2$ for each $n \in \mathbb{N}$, and then, passing to the limits, we get that $\lim_{n\to\infty}|\theta(x_n)| = 0$. Note by Lemma 2.4(iii) that θ is continuous and so $\theta(\bar{x}) = 0$. Then, by Lemma 2.4(ii), one has that $\bar{x}$ is a critical point, and thus, it is a local Pareto optimum (by (2.2)). Fix $n \in \mathbb{N}$. One has by (3.5) that $\|x_{n+l}-x_n\| \le t_{n+l}-t_n$ for each $l \in \mathbb{N}$, and so (3.6) is seen to hold by passing to the limits (as $l \to \infty$).

Finally, we prove the quadratic convergence rate of $\{x_n\}$ to $\bar{x}$. Fix $n \in \mathbb{N}$, and note from (3.5) and implication (3.10) that

$$\|s(x_{n+j})\| \le (t_{n+j+1}-t_{n+j})\left(\frac{\|s(x_n)\|}{t_{n+1}-t_n}\right)^2 \quad \text{for each } j \in \mathbb{N}. \quad (3.16)$$

In view of Algorithm 3.1, one sees that $\|x_i - x_{n+1}\| \le \sum_{j=n+1}^{i-1}\|s(x_j)\|$ for each $i > n+1$. Letting $i \to \infty$, one has by the convergence of $\{x_n\}$ to $\bar{x}$ and by (3.16) that

$$\|\bar{x}-x_{n+1}\| \le \sum_{j=n+1}^{\infty}\|s(x_j)\| \le (r_a^*-t_{n+1})\left(\frac{\|s(x_n)\|}{t_{n+1}-t_n}\right)^2 \le \frac{r_a^*-t_{n+1}}{t_{n+1}-t_n}\|s(x_n)\| \quad (3.17)$$

(by (3.5)). Then, it follows that

$$\|\bar{x}-x_n\| \ge \|x_{n+1}-x_n\| - \|\bar{x}-x_{n+1}\| \ge \frac{2t_{n+1}-t_n-r_a^*}{t_{n+1}-t_n}\|s(x_n)\|. \quad (3.18)$$

By assumption that $\beta < b_a$, Proposition 2.6(iii) is applicable, and then we have by the equality of (2.17) that there exists $N \in \mathbb{N}$ such that

$$\frac{2t_{n+1}-t_n-r_a^*}{t_{n+1}-t_n} > 0 \quad \text{for each } n \ge N.$$

Therefore, combining (3.17) and (3.18), we obtain (3.7). This, together with the inequality of (2.17), ensures the quadratic convergence rate of $\{x_n\}$ to $\bar{x}$. The proof is complete. □

Theorem 3.5 below shows that under almost the same conditions as in Theorem 3.4, a sequence $\{x_n\}$ generated by Algorithm 1.1 or 3.2 with initial point x_0 is the one generated by Algorithm 3.1 with the same initial point x_0. Hence, all the conclusions of Theorem 3.4 hold for Algorithm 1.1 or 3.2.

THEOREM 3.5. *Suppose that F satisfies assumption* (3.3) *associated to* $(x_0; a, r_a)$ *and* L, *and*

$$\|s(x_0)\| \le \beta \le \frac{3(1-\sigma)\left(1 - a\int_0^{r_a^*} L(u)\mathrm{d}u\right)}{aL(r_a^*)}. \tag{3.19}$$

Then, with initial point x_0, any sequence $\{x_n\}$ generated by Algorithm 3.2 coincides with the one generated by Algorithm 3.1; consequently, the conclusions of Theorem 3.4 hold.

Proof. Below, we only show the case when $\{x_n\}$ is a sequence generated by Algorithm 1.1 with initial point x_0 because the proof is similar for Algorithm 3.2. To finish the proof of this theorem, fix $i \in \mathbb{N}$. First, we show the following implication:

$$[\|s(x_i)\| \le t_{i+1} - t_i,\ \|x_i - x_0\| + \|s(x_i)\| \le r_a^*] \Rightarrow [x_{i+1} = x_i + s(x_i)]. \tag{3.20}$$

For this purpose, we assume that

$$\|s(x_i)\| \le t_{i+1} - t_i \quad \text{and} \quad \|x_i - x_0\| + \|s(x_i)\| \le r_a^*. \tag{3.21}$$

Noting by Proposition 2.6 that $r_a^* \le r_a$, we have $x_i \in \mathbf{B}(x_0, r_a)$, and then obtain from Lemma 3.3 and (2.14) that for each $\lambda \in \mathbf{\Delta}_m$, $\nabla^2 F_\lambda(x_i)$ is positive definite and

$$\|\nabla^2 F_\lambda(x_i)^{-1}\| \le -ah_a'(\|x_i - x_0\|)^{-1}. \tag{3.22}$$

By assumption (3.21), one has $x_i + s(x_i) \in \mathbf{B}(x_0, r_a)$. Fix $j \in \{1, \dots, m\}$. By the Taylor formula, one has that

$$\begin{aligned}
&F_j(x_i + s(x_i)) \\
&= F_j(x_i) + \nabla F_j(x_i)^T s(x_i) + \tfrac{1}{2} s(x_i)^T \nabla^2 F_j(x_i) s(x_i) \\
&\quad + \textstyle\int_0^1 s(x_i)^T (\nabla^2 F_j(x_i + ts(x_i)) - \nabla^2 F_j(x_i)) s(x_i)(1-t)\mathrm{d}t \\
&\le F_j(x_i) + \nabla F_j(x_i)^T s(x_i) + \tfrac{1}{2} s(x_i)^T \nabla^2 F_j(x_i) s(x_i) + \tfrac{L(r_a^*)}{6} \|s(x_i)\|^3,
\end{aligned}$$

where the inequality holds because

$$\|\nabla^2 F_j(x_i + ts(x_i)) - \nabla^2 F_j(x_i)\| \le \int_{\|x_i - x_0\|}^{\|x_i - x_0\| + t\|s(x_i)\|} L(u)\mathrm{d}u \le L(r_a^*)\|s(x_i)\| t$$

(due to assumption (3.3) and the fact that $L(\cdot)$ is nondecreasing and positive). By the definition of θ (cf. (2.5)), this implies that

$$\begin{aligned}
F_j(x_i + s(x_i)) &\le F_j(x_i) + \theta(x_i) + \tfrac{L(r_a^*)}{6} \|s(x_i)\|^3 \\
&= F_j(x_i) + \sigma\theta(x_i) + (1-\sigma)\theta(x_i) + \tfrac{L(r_a^*)}{6} \|s(x_i)\|^3,
\end{aligned} \tag{3.23}$$

where $\sigma \in (0,1)$ is the parameter in Algorithm 1.1. Recall from (2.9) and Lemma 2.4(i) that

$$\theta(x_i) \le -\frac{\rho_{\min}(\lambda, x_i)}{2}\|s(x_i)\|^2. \tag{3.24}$$

Recalling by (2.8) that $\rho_{\min}(\lambda, x_i) = \|\nabla^2 F_\lambda(x_i)^{-1}\|^{-1}$, it follows from (3.22) and (3.24) that

$$\theta(x_i) \le \frac{1}{2a}h'_a(\|x_i - x_0\|)\|s(x_i)\|^2 \le \frac{1}{2a}h'_a(r_a^*)\|s(x_i)\|^2, \tag{3.25}$$

where the last inequality holds because that $h'(\cdot)$ is monotonically increasing on $[0, r_a^*]$. Note that $\{t_{i+1} - t_i\}$ is monotonically decreasing (cf [29, Lemma 2.4]), and so, for each $i \in \mathbb{N}$, $t_{i+1} - t_i \le t_1 - t_0 = \beta$ (by (2.15)). This, together with (3.21), implies that

$$\|s(x_i)\| \le t_{i+1} - t_i \le t_1 - t_0 = \beta \le \frac{3(1-\sigma)\left(1 - a\int_0^{r_a^*} L(u)\mathrm{d}u\right)}{aL(r_a^*)} = \frac{-3(1-\sigma)h'_a(r_a^*)}{aL(r_a^*)},$$

where the last inequality is due to (3.19). Combining this with (3.25) yields that

$$(1-\sigma)\theta(x_i) + \frac{L(r_a^*)\|s(x_i)\|}{6}\|s(x_i)\|^2 \le \left(\frac{L(r_a^*)\|s(x_i)\|}{3} + \frac{(1-\sigma)h'_a(r_a^*)}{a}\right)\frac{\|s(x_i)\|^2}{2} \le 0;$$

then, (3.23) implies that

$$F_j(x_i + s(x_i)) \le F_j(x_i) + \sigma\theta(x_i) \quad \text{for all } j = 1, \ldots, m.$$

Thus, in view of Algorithm 1.1, we have $x_{i+1} = x_i + s(x_i)$ and so (3.20) is seen to hold.

Below, we show by induction that $\{x_n\}$ coincides with the sequence generated by Algorithm 3.1 with the same initial point x_0, namely the following assertion holds for each $n \in \{0\} \cup \mathbb{N}$:

$$x_{n+1} = x_n + s(x_n) \tag{3.26}$$

Since $\|s(x_0)\| \le \beta = t_1 - t_0 \le r_a^*$ by (3.19) and Proposition 2.6(i), it follows from (3.20) that (3.26) holds for $n = 0$. Suppose that $x_1, \ldots, x_k$ are the same points as generated by Algorithm 3.1. Then, by Theorem 3.4, we have that $x_i \in \mathbf{B}(x_0, r_a^*)$ and $\|s(x_i)\| \le t_{i+1} - t_i$ for $i = 1, \ldots, k$, and

$$\|x_k - x_0\| + \|s(x_k)\| \le \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| + \|s(x_k)\| \le t_{k+1} < r_a^*.$$

This implies that the assumptions of implication (3.20) hold when $i = k$. Then, it follows from implication (3.20) that $\alpha_k = 1$, and so, (3.26) holds for $n = k$. Thus, x_{k+1} is the same point as generated by Algorithm 3.1. Then, we obtain inductively that $\{x_n\}$ is same as the sequence generated by Algorithm 3.1 with same initial point x_0. Therefore, the conclusions of Theorem 3.4 hold and the proof is complete. □

3.2. Estimation of convergence radius. This subsection is devoted to providing an estimate of the radius of the convergence ball of the extended Newton method (without or with line-search scheme) for multiobjective optimization under the L-average Lipschitz condition. For this purpose, let $a^* > 0$ be such that (2.11) is reduced to

$$\frac{1}{R}\int_0^R L(u)(R-u)\mathrm{d}u > \frac{1}{a^*}. \tag{3.27}$$

Let (r_{a^*}, b_{a^*}) be the pair of positive constants given by (2.12) with a^* in place of a. Let $x^* \in U$ be a local Pareto optimum of F, and assume that F satisfies assumption (3.3) associated to $(x^*; a^*, r_{a^*})$ and L. Throughout this subsection, we always assume that $L(\cdot)$ is left-hand continuous. Write

$$\xi^* := \max\{\|\nabla^2 F_i(x^*)\| : i = 1, \ldots, m\}. \tag{3.28}$$

A useful proposition is as follows.

Proposition 3.6. *Suppose that F satisfies assumption* (3.3) *associated to* $(x^*; a^*, r_{a^*})$ *and* L. *Let* $x_0 \in \mathbf{B}\left(x^*, \frac{b_{a^*}}{1+a^*\xi^*}\right)$. *Then, the following assertions hold:*

(i) *F satisfies assumption* (3.3) *associated to* $(x_0; \bar{a}, \bar{r})$ *and* $\bar{L}$ *given by*

$$\bar{a} := \frac{a^*}{1 - a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u}, \quad \bar{r} := r_{a^*} - \|x_0 - x^*\| \tag{3.29}$$

and

$$\bar{L}(u) := L(\|x_0 - x^*\| + u) \quad \textit{for each } u \in [0, R - \|x_0 - x^*\|); \tag{3.30}$$

(ii) *$s(x_0)$ satisfies that*

$$\|s(x_0)\| \le \frac{a^*\int_0^{\|x_0-x^*\|} L(u)(\|x_0 - x^*\| - u)\mathrm{d}u + a^*\xi^*\|x_0 - x^*\|}{1 - a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u}. \tag{3.31}$$

Proof. (i) Write $\bar{R} := R - \|x_0 - x^*\|$. We first show (2.11) holds with $\bar{a}, \bar{R}, \bar{L}$ in place of a, R, L. By definition of r_{a^*} in (2.12) (applied to a^* in place of a), one has

$$a^*\int_0^{r_{a^*}} L(u)\mathrm{d}u = 1. \tag{3.32}$$

Thus, it suffices to show that

$$\int_{\|x_0-x^*\|}^R L(u)(R-u)\mathrm{d}u \ge (R - \|x_0 - x^*\|)\int_{\|x_0-x^*\|}^{r_{a^*}} L(u)\mathrm{d}u, \tag{3.33}$$

thanks to the definitions of $\bar{a}, \bar{R}, \bar{L}$. To do this, by (3.27), one has

$$\begin{aligned}\int_{\|x_0-x^*\|}^R L(u)(R-u)\mathrm{d}u &\ge \tfrac{R}{a^*} - \int_0^{\|x_0-x^*\|} L(u)(R-u)\mathrm{d}u \\ &= R\int_{\|x_0-x^*\|}^{r_{a^*}} L(u)\mathrm{d}u + \int_0^{\|x_0-x^*\|} L(u)u\mathrm{d}u,\end{aligned}$$

where the equality holds because, by (3.32),

$$\frac{R}{a^*} = R\int_0^{r_{a^*}} L(u)\mathrm{d}u = R\int_{\|x_0-x^*\|}^{r_{a^*}} L(u)\mathrm{d}u + R\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u.$$

Hence, (3.33) is seen to hold, showing (2.11) (with $\bar{a}, \bar{R}, \bar{L}$ in place of a, R, L), namely the first assumption in (3.3) (associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$). To show the second assumption in (3.3), noting first that $\|x_0 - x^*\| < \frac{b_{a^*}}{1+a^*\xi^*} < b_{a^*} < r_{a^*}$, Lemma 3.3 is applicable to concluding that, for each $j = 1, \ldots, m$, $\nabla^2 F_j(x_0)$ is positive definite, and

$$\|\nabla^2 F_j(x_0)^{-1}\| \le \frac{\|\nabla^2 F_j(x^*)^{-1}\|}{1 - a^* \int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u} \le \frac{a^*}{1 - a^* \int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u} = \bar{a};$$

consequently, the second assumption in (3.3) (associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$) is checked. Now let us verify the last assumption in (3.3) (associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$). To do this, let $x, y \in \mathbf{B}(x_0, \bar{r})$ be such that $\|x - x_0\| + \|y - x\| < \bar{r}$, and fix j. Then,

$$\|x - x^*\| + \|y - x\| \le \|x_0 - x^*\| + \|x - x_0\| + \|y - x\| \le \|x_0 - x^*\| + \bar{r} = r_{a^*}.$$

Thus, it follows from the last assumption in (3.3) (associated to $(x^*; a^*, r_{a^*})$ and L) that

$$\|\nabla^2 F_j(y) - \nabla^2 F_j(x)\| \le \int_{\|x_0-x^*\|+\|x_0-x\|}^{\|x_0-x^*\|+\|x_0-x\|+\|x-y\|} L(u)\mathrm{d}u = \int_{\|x_0-x\|}^{\|x_0-x\|+\|x-y\|} \bar{L}(u)\mathrm{d}u.$$

This shows the third assumption in (3.3) (associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$) and the proof for assertion (i) is complete.

(ii) Noting that x^* is a local Pareto optimum of F, we obtain from (2.2) that x^* is a critical point of F. Therefore, it follows from Lemma 2.4 that $s(x^*) = 0$. Note by (2.6) that there exists $\lambda(:= \lambda(x^*)) \in \mathbf{\Delta}_m$ (the KKT multipliers of problem (2.3)) such that

$$s(x^*) = -\left[\sum_{j=1}^m \lambda_j(x^*)\nabla^2 F_j(x^*)\right]^{-1} \sum_{j=1}^m \lambda_j(x^*)\nabla F_j(x^*) = -\nabla^2 F_\lambda(x^*)^{-1}\nabla F_\lambda(x^*),$$

where F_λ is given by (2.7). Hence $\nabla F_\lambda(x^*) = 0$, and

$$\begin{aligned}\|\nabla F_\lambda(x_0) - \nabla^2 F_\lambda(x^*)(x_0 - x^*)\| &= \| \textstyle\int_0^1 (\nabla^2 F_\lambda(x^* + \tau(x_0 - x^*)) - \nabla^2 F_\lambda(x^*))(x_0 - x^*)\mathrm{d}\tau\| \\ &\le \int_0^1 \int_0^{\|x_0-x^*\|\tau} L(u)\|x_0 - x^*\|\mathrm{d}u\mathrm{d}\tau \\ &= \int_0^{\|x_0-x^*\|} L(u)(\|x_0 - x^*\| - u)\mathrm{d}u,\end{aligned}$$

thanks to the third assumption in (3.3) (associated to $(x^*; a^*, r_{a^*})$). Therefore,

$$\begin{aligned}a^*\|\nabla F_\lambda(x_0)\| &\le a^*\|\nabla F_\lambda(x_0) - \nabla^2 F_\lambda(x^*)(x_0 - x^*)\| + a^*\|\nabla^2 F_\lambda(x^*)\|\|x_0 - x^*\| \\ &\le a^* \int_0^{\|x_0-x^*\|} L(u)(\|x_0 - x^*\| - u)\mathrm{d}u + a^*\xi^*\|x_0 - x^*\|.\end{aligned} \tag{3.34}$$

Furthermore, by Lemma 3.3, one has that

$$\|\nabla^2 F_\lambda(x_0)^{-1}\| \le \frac{\|\nabla^2 F_\lambda(x^*)^{-1}\|}{1-a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u} \le \frac{a^*}{1-a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u}.$$

This, together with (2.10) and (3.34), implies that

$$\|s(x_0)\| \le \|\nabla^2 F_\lambda(x_0)^{-1}\|\|\nabla F_\lambda(x_0)\| \le \frac{a^*\int_0^{\|x_0-x^*\|} L(u)(\|x_0-x^*\|-u)\mathrm{d}u + a^*\xi^*\|x_0-x^*\|}{1-a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u}.$$

The proof is complete. □

THEOREM 3.7. *Suppose that F satisfies assumption* (3.3) *associated to* $(x^*; a^*, r_{a^*})$ *and* L. *Let* $x_0 \in \mathbf{B}(x^*, \frac{b_{a^*}}{1+a^*\xi^*})$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 3.1 with initial point* x_0 *is well-defined and converges quadratically to a local Pareto optimum of* F.

Proof. By Proposition 3.6, F satisfies assumption (3.3) associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$ defined by (3.29) and (3.30), respectively. Let

$$\bar{\beta} := \frac{a^*\int_0^{\|x_0-x^*\|} L(u)(\|x_0-x^*\|-u)\mathrm{d}u + a^*\xi^*\|x_0-x^*\|}{1-a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u}. \tag{3.35}$$

Then $\|s(x_0)\| \le \bar{\beta}$ (by (3.31)). Thus, to apply Theorem 3.4 with $\bar{\beta}, \bar{a}, \bar{L}$, in place of β, a, L, we have to show that

$$\bar{\beta} < \bar{b}_{\bar{a}} \quad \text{and} \quad \bar{r}^*_{\bar{a}} \le \bar{r}, \tag{3.36}$$

where $\bar{r}^*_{\bar{a}}$ and $\bar{b}_{\bar{a}}$ denote respectively the corresponding r^*_a and b_a given by (2.16) and (2.12) with $\bar{\beta}, \bar{a}, \bar{L}$, in place of β, a, L. To do this, write $\tau := \|x_0 - x^*\|$ for simplicity. Let $\bar{r}_{\bar{a}}$ be the corresponding r_a defined by (2.12) with $\bar{a}, \bar{L}$ in place of a, L. Then,

$$\frac{a^*\int_0^{\bar{r}_{\bar{a}}} L(\|x_0-x^*\|+u)\mathrm{d}u}{1-a^*\int_0^{\|x_0-x^*\|} L(u)\mathrm{d}u} = \bar{a}\int_0^{\bar{r}_{\bar{a}}} L(\tau+u)\mathrm{d}u = 1 \tag{3.37}$$

by the definition of $\bar{a}$ (see (3.29)). By the definition of r_{a^*} (see (3.32)), it follows from (3.37) that

$$a^*\int_0^{\bar{r}_{\bar{a}}} L(\tau+u)\mathrm{d}u = 1 - a^*\int_0^{\tau} L(u)\mathrm{d}u = a^*\int_0^{r_{a^*}} L(u)\mathrm{d}u - a^*\int_0^{\tau} L(u)\mathrm{d}u = a^*\int_{\tau}^{r_{a^*}} L(u)\mathrm{d}u;$$

hence

$$\int_{\tau}^{\bar{r}_{\bar{a}}+\tau} L(u)\mathrm{d}u = \int_{\tau}^{r_{a^*}} L(u)\mathrm{d}u. \tag{3.38}$$

Since $a^* > 0$ and $L(\cdot)$ is positive and nondecreasing, it follows from (3.38) and the definition of τ that

$$\bar{r}_{\bar{a}} + \|x_0 - x^*\| = \bar{r}_{\bar{a}} + \tau = r_{a^*}. \tag{3.39}$$

This, together with the definition of $\bar{b}_{\bar{a}}$, implies that

$$\bar{b}_{\bar{a}} = \bar{a}\int_0^{\bar{r}_{\bar{a}}} \bar{L}(u)u\mathrm{d}u = \bar{a}\int_0^{r_{a^*}-\|x_0-x^*\|} L(\|x_0-x^*\|+u)u\mathrm{d}u = \bar{a}\int_{\|x_0-x^*\|}^{r_{a^*}} L(u)(u-\|x_0-x^*\|)\mathrm{d}u.$$

Note also by the definition of $\bar{\beta}$ in (3.35) that

$$\bar{\beta} = \bar{a}\left(\int_0^{\|x_0-x^*\|} L(u)(\|x_0-x^*\|-u)\mathrm{d}u + \xi^*\|x_0-x^*\|\right).$$

Therefore, $\bar{\beta} < \bar{b}_{\bar{a}}$ if and only if

$$\xi^*\|x_0-x^*\| < \int_0^{r_{a^*}} L(u)(u-\|x_0-x^*\|)\mathrm{d}u = \frac{b_{a^*}}{a^*} - \frac{\|x_0-x^*\|}{a^*}$$

(noting that $a^*\int_0^{r_{a^*}} L(u)\mathrm{d}u = 1$ and $b_{a^*} = a^*\int_0^{r_{a^*}} L(u)u\mathrm{d}u$ by definition), which holds by the assumption that $0 < \|x_0-x^*\| < \frac{b_{a^*}}{1+a^*\xi^*}$. Hence, $\bar{\beta} < \bar{b}_{\bar{a}}$ and so, by (2.16) (with $\bar{\beta}, \bar{a}, \bar{L}$, in place of β, a, L), one has $\bar{r}^*_{\bar{a}} \le \bar{r}_{\bar{a}} = r_{a^*} - \|x_0-x^*\| = \bar{r}$ (by (3.39)). Consequently, (3.36) is proved and the proof is complete. □

For the following theorem, we need some more notations and an additional lemma. Fix $\tau \in (0, r_{a^*})$, and set

$$a_\tau := \frac{a^*}{1-a^*\int_0^\tau L(u)\mathrm{d}u} \quad \text{and} \quad \beta_\tau := \frac{a^*\int_0^\tau L(u)(\tau-u)\mathrm{d}u + a^*\xi^*\tau}{1-a^*\int_0^\tau L(u)\mathrm{d}u}. \tag{3.40}$$

Let $\bar{h}_{a_\tau}(\cdot)$ be the majorizing function given by (2.13) with $\beta_\tau, a_\tau, L(\tau+\cdot)$ in place of $\beta, a, L(\cdot)$, that is,

$$\bar{h}_{a_\tau}(t) := \beta_\tau - t + a_\tau\int_0^t L(\tau+u)(t-u)\mathrm{d}u \quad \text{for each } t \in [0, R-\tau).$$

Lemma 3.8. *Let $\tau \in \left(0, \frac{b_{a^*}}{1+a^*\xi^*}\right)$. Then, $\bar{h}_{a_\tau}$ has two zeroes on $[0, R-\tau)$, and there exists $r \in \left(0, \frac{b_{a^*}}{1+a^*\xi^*}\right)$ such that*

$$\beta_\tau \le \frac{3(1-\sigma)\left(1-a_\tau\int_0^{\bar{r}^*_{a_\tau}} L(\tau+u)\mathrm{d}u\right)}{a_\tau L(\tau+\bar{r}^*_{a_\tau})} \quad \textit{for each } \tau \in (0, r), \tag{3.41}$$

where $\bar{r}^*_{a_\tau}$ is the smaller zero of $\bar{h}_{a_\tau}$ on $[0, R-\tau)$.

Proof. Let $\bar{r}_{a_\tau}$ and $\bar{b}_{a_\tau}$ denote respectively the corresponding r_a and b_a given by (2.12) with $\beta_\tau, a_\tau, L(\tau+\cdot)$ in place of $\beta, a, L(\cdot)$. Then as we did for proving (3.39) and that $\bar{\beta} < \bar{b}_{\bar{a}}$ in the proof of Theorem 3.7 (cf. (3.36)), we can verify that

$$\beta_\tau < \bar{b}_{a_\tau} \quad \text{and} \quad \bar{r}_{a_\tau} = r_{a^*} - \tau. \tag{3.42}$$

Thus, Proposition 2.6 is applicable (to $\beta_\tau, a_\tau, L(\tau+\cdot)$ in place of $\beta, a, L(\cdot)$) to concluding that $\bar{h}_{a_\tau}$ has two zeroes on $[0, R-\tau)$, and the proof of the first assertion is complete. To show the second assertion, note by definition that $\lim_{\tau\to 0^+}\beta_\tau = 0$, $\lim_{\tau\to 0^+} a_\tau = a^*$, $\lim_{\tau\to 0^+}\bar{b}_{a_\tau} = b_{a^*}$(see (3.40) and (2.12)), and that $\lim_{\tau\to 0^+}\bar{r}_{a_\tau} = r_{a^*}$ by (3.42). Hence, $\lim_{\tau\to 0^+}\bar{r}^*_{a_\tau} = 0$ thanks to (2.16) (applied to $\beta_\tau, a_\tau, L(\tau+\cdot)$ in place of β, a, $L(\cdot)$) and so $\lim_{\tau\to 0^+} a_\tau\int_0^{\bar{r}^*_{a_\tau}} L(\tau+u)\mathrm{d}u = 0$. Thus, it follows from the assumed left-hand continuity assumption for L that

$$\lim_{\tau\to 0^+}\frac{3(1-\sigma)\left(1-a_\tau\int_0^{\bar{r}^*_{a_\tau}} L(\tau+u)\mathrm{d}u\right)}{a_\tau L(\tau+\bar{r}^*_{a_\tau})} \geq \frac{3(1-\sigma)}{a^*L(r^*_a)} > 0.$$

Since $\lim_{\tau\to 0^+}\beta_\tau = 0$ and the function $\tau\mapsto\beta_\tau$ is continuous on $[0, r_{a^*})$, it follows that there exists $0 < r \leq \frac{b_{a^*}}{1+a^*\xi^*}$ satisfying (3.41), and the proof is complete. □

Theorem 3.9 below shows that if F satisfies assumption (3.3) associated to $(x^*; a^*, r_{a^*})$ and L, then there exists $r > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 1.1 or 3.2 with initial point $x_0 \in \mathbf{B}(x^*, r)$ converges quadratically to a local Pareto optimum of F. In the next section, we provide an explicitly estimate of the radius r for the special case when $L(\cdot)$ is a constant function.

THEOREM 3.9. *Suppose that F satisfies assumption* (3.3) *associated to* $(x^*; a^*, r_{a^*})$ *and* L. *Let* $r \in \left(0, \frac{b_{a^*}}{1+a^*\xi^*}\right)$ *satisfy* (3.41), *and let* $x_0 \in \mathbf{B}(x^*, r)$. *Then, any sequence* $\{x_n\}$ *generated by Algorithms 3.2 with initial point* x_0 *converges quadratically to a local Pareto optimum of* F.

Proof. Note by assumption that $x_0 \in \mathbf{B}\left(x^*, \frac{b_{a^*}}{1+a^*\xi^*}\right)$. The proof is similar to that for Theorem 3.7. Indeed, let $\bar{\beta}$, $\bar{a}$, $\bar{r}$, $\bar{r}_{\bar{a}}$, $\bar{r}^*_{\bar{a}}$ and $\bar{L}$ be as in the proof of Theorem 3.7. Then, one has that $\bar{r}_{\bar{a}} = \bar{r}$ (by (3.39)), and that $\|s(x_0)\| \leq \bar{\beta}$ and F satisfies assumption (3.3) associated to $(x_0; \bar{a}, \bar{r})$ and $\bar{L}$ (by Proposition 3.6) and so to $(x_0; \bar{a}, \bar{r}_{\bar{a}})$. Thus, by Theorem 3.5 (applied to $\bar{\beta}, \bar{a}, \bar{L}$ in place of β, a, L), it suffices to show that

$$\bar{\beta} \leq \frac{3(1-\sigma)\left(1-\bar{a}\int_0^{\bar{r}^*_{\bar{a}}}\bar{L}(u)\mathrm{d}u\right)}{\bar{a}\bar{L}(\bar{r}^*_{\bar{a}})}. \tag{3.43}$$

To do this, we write $\tau := \|x_0 - x^*\|$ for simplicity. Then, one has by definition that $\beta_\tau = \bar{\beta}$, $a_\tau = \bar{a}$ and $\bar{r}^*_{a_\tau} = \bar{r}^*_{\bar{a}}$, where β_τ and a_τ are defined by (3.40). Since $\tau = \|x_0 - x^*\| < r$ by assumption, (3.43) follows from (3.41) because $\bar{L}(\cdot) = L(\|x_0 - x^*\| + \cdot) = L(\tau + \cdot)$, and the proof is complete. □

4. Applications. By virtue of the results established in the preceding section, this section is devoted to establishing convergence analysis theorems under the classical Lipschitz condition or the γ-condition for multiobjective optimization. In particular, the global convergence of Algorithm 3.2 is established under the classical Lipschitz condition.

4.1. Theorems under the classical Lipschitz condition and global version of the extended Newton method with its convergence.

4.1.1. Theorems under the classical Lipschitz condition. Kantorovich's theorem [26] is one of the famous results on the Newton method, which provides a criterion for ensuring its quadratic convergence under the classical Lipschitz condition. The main point of Kantorovich's type premise is to let $L(\cdot)$ mentioned in the preceding section be a constant function. In this case, the L-average Lipschitz condition of $\nabla^2 F_j$ is reduced to the classical Lipschitz condition of $\nabla^2 F_j$ for each $j = 1, \ldots, m$. That is, there are $L > 0$ and $r > 0$ such that

$$\|\nabla^2 F_j(x) - \nabla^2 F_j(y)\| \leq L\|x - y\| \quad \text{for each } x, y \in \mathbf{B}(x_0, r).$$

Then the function $L(\tau + \cdot)$ is independent of the choice of τ and coincides with L, that is, $L(\tau + \cdot) = L(\cdot) = L$ on $\mathbb{R}^+$. Thus, for any $a > 0$, one has that

$$b_a = \frac{1}{2aL}, \quad r_a = \frac{1}{aL},$$

and the majorizing functions h_a defined by (2.13) is reduced to

$$h_a(t) = \beta - t + \frac{aL}{2}t^2 \quad \text{for each } t \in \mathbb{R}.$$

Therefore, if $\beta \leq \frac{1}{2aL}$, one has by (2.12), (2.15) and (2.16) (see also [42]) that

$$r_a^* = \frac{1 - \sqrt{1 - 2aL\beta}}{aL}, \tag{4.1}$$

$$t_{a,n} = \frac{1 - q_a^{2^n - 1}}{1 - q_a^{2^n}} r_a^* \quad \text{and} \quad t_{a,n+1} - t_{a,n} = \frac{1 - q_a}{1 - q_a^{2^{n+1}}} q_a^{2^n - 1} r_a^* \quad \text{for each } n \in \mathbb{N},$$

where

$$q_a := \frac{1 - \sqrt{1 - 2aL\beta}}{1 + \sqrt{1 - 2aL\beta}}, \tag{4.2}$$

and we adopt the convention that $\frac{1 - q_a^{2^n - 1}}{1 - q_a^{2^n}} := 1 - (\frac{1}{2})^n$ and $\frac{1 - q_a}{1 - q_a^{2^{n+1}}} := \left(\frac{1}{2}\right)^{n+1}$ if $q_a = 1$.

Theorem 4.1 follows directly from Theorems 3.4 and 3.5, and establishes a quantitative convergence criterion of the extended Newton method for multiobjective optimization under the classical Lipschitz condition.

THEOREM 4.1. *Suppose that* $\|s(x_0)\| \leq \beta$ *and* F *satisfies assumption* (3.3) *associated to* $(x_0; a, r_a^*)$ *and* $L(\cdot) \equiv L$. *Let* q_a *be given by* (4.2). *Then, with initial point* x_0, *we have the following assertions:*

(i) *If* $\beta \leq \frac{1}{2aL}$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 is well-defined, stays in* $\mathbf{B}(x_0, r_a^*)$, *and converges to a local Pareto optimum* $\bar{x} \in \mathbf{B}[x_0, r_a^*]$ *with the following error estimates:*

$$\|x_{n+1} - x_n\| \leq \frac{1 - q_a}{1 - q_a^{2^{n+1}}} q_a^{2^n - 1} r_a^* \quad \text{and} \quad \|x_n - \bar{x}\| \leq \frac{1 - q_a}{1 - q_a^{2^n}} q_a^{2^n - 1} r_a^* \quad \text{for each } n \in \mathbb{N}. \tag{4.3}$$

(ii) *If* $\beta < \frac{1}{2aL}$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges quadratically to* $\bar{x}$ *with the following error estimate for some* $N \in \mathbb{N}$:

$$\|x_{n+1} - \bar{x}\| \leq \frac{q_a(1 - q_a^{2^{n+1}})}{(1 - q_a)(1 - q_a^{2^n})^2 r_a^*} \|x_n - \bar{x}\|^2 \quad \text{for each } n \geq N. \tag{4.4}$$

(iii) *If* $\beta \le \frac{-9(1-\sigma)^2+3(1-\sigma)\sqrt{1+9(1-\sigma)^2}}{aL}$, *then* $\beta < \frac{1}{2aL}$, *and any sequence* $\{x_n\}$ *generated by Algorithm 3.2 coincides with the one generated by Algorithm 3.1, and satisfies* (4.3) *and* (4.4).

Proof. Assertions (i) and (ii) follow directly from Theorem 3.4. Then, it remains to show assertion (iii). In fact, assume that $\beta \le \frac{-9(1-\sigma)^2+3(1-\sigma)\sqrt{1+9(1-\sigma)^2}}{aL}$. Then $\beta < \frac{1}{2aL}$ because $-9(1-\sigma)^2 + 3(1-\sigma)\sqrt{1+9(1-\sigma)^2} < \frac{1}{2}$. Since $L(\cdot) \equiv L$, it follows from (4.1) that $\frac{3(1-\sigma)\left(1-a\int_0^{r_a^*} L(u)\mathrm{d}u\right)}{aL(r_a^*)} = \frac{3(1-\sigma)\sqrt{1-2aL\beta}}{aL}$. Thus, (3.19) holds because it is equivalent that $aL\beta \le 3(1-\sigma)\sqrt{1-2aL\beta}$, which is true by assumption. Hence, the conclusion follows from Theorem 3.5. □

REMARK 4.1. Under the assumption made in Theorem 4.1, we see that there exist $V \subseteq \mathbf{B}(x_0, r_a^*)$, $\bar{a} := \frac{1}{a}$ and $\bar{b} > 0$ such that $\bar{a}\mathrm{I} \le \nabla^2 F_j(x) \le \bar{b}\mathrm{I}$ for all $x \in V$ and all $j = 1, \ldots, m$, where, for A, $B \in \mathbb{R}^{n\times n}$, $A \ge B$ means that $A - B$ is positive semi-definite. Thus [22, Theorem 6.1] could apply. However, Theorem 4.1 cannot be derived via a direct application of [22, Theorem 6.1]. In fact, Example 4.1 below illustrates the case where Theorem 4.1 is applicable but not [22, Theorem 6.1].

EXAMPLE 4.1. Let $\sigma \in (\frac{1}{2}, 1)$ and let τ satisfy

$$(1-\sigma)\sigma < \tau \le -9(1-\sigma)^2 + 3(1-\sigma)\sqrt{1+9(1-\sigma)^2}. \tag{4.5}$$

Consider problem (1.1) with $m = l = 1$ and $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := -\tau x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \quad \text{for each } x \in \mathbb{R}.$$

Then

$$F''(x) = 1 - x \quad \text{for each } x \in \mathbb{R}. \tag{4.6}$$

Let $x_0 = 0$. Then, one checks that

$$a := \|F''(x_0)^{-1}\| = 1, \quad \|s(x_0)\| = \| - (F''(x_0))^{-1}F'(x_0)\| = \tau, \tag{4.7}$$

and F'' satisfies the Lipschitz condition with modulus $L = 1$ on $[-1, 1]$. By (4.5), we see that Theorem 4.1(iii) is applicable, and we can conclude that any sequence $\{x_n\}$ generated by Algorithm 3.2 (and so Algorithm 1.1) with initial point x_0 converges to a local Pareto optimum. We show in the below that [22, Theorem 6.1] is not applicable. To do this, suppose on the contrary that [22, Theorem 6.1] is applicable. Then, there exist $0 < r < 1$ and positive numbers $a_r, b_r, \delta, \varepsilon$ such that

$$\frac{\varepsilon}{a_r} \le 1 - \sigma, \quad \|s(x_0)\| \le \min\left\{\delta, r(1 - \frac{\varepsilon}{a_r})\right\}, \quad a_r \le F''(x) \le b_r \text{ for all } x \in (-r, r), \tag{4.8}$$

and $\|F''(x) - F''(y)\| \le \varepsilon$ for all $x, y \in (-r, r)$ with $\|x - y\| \le \delta$. Then, by (4.6), without loss of generality, we take $a_r = 1 - r$ and $\delta = \varepsilon \le (1 - r)(1 - \sigma)$. Thus, if $r \ge 1 - \sigma$, one has that $\|s(x_0)\| \le \delta \le \sigma(1 - \sigma)$. Below we shows that this is also true if $r \le 1 - \sigma$. Granting this, one has from (4.7) that $\tau \le \sigma(1 - \sigma)$, which is a contradiction to (4.5). To proceed, assume $r \le 1-\sigma$, and note that the function $t \mapsto \min\{t, r(1-\frac{t}{1-r})\}$ attains its maximum t_0 on $[0,(1-r)(1-\sigma)]$ at t_0 satisfying $t_0 = r(1-\frac{t_0}{1-r})$, i.e., $t_0 = r(1-r)$. Since $\sigma \in (\frac{1}{2}, 1)$ by assumption, it follows that $r \le 1-\sigma \le \frac{1}{2}$ and so $\min\{\delta, r(1-\frac{\delta}{1-r})\} \le t_0 = r(1-r) \le \sigma(1-\sigma)$. Thus we have by (4.8) that $\|s(x_0)\| \le \min\{\delta, r(1-\frac{\delta}{1-r})\} \le \sigma(1-\sigma)$, as desired to show.

Theorem 4.2 below follows directly from Theorems 3.7 and 3.9, and provides explicit estimates of the convergence radius of the extended Newton method for multiobjective optimization under the classical Lipschitz condition. In particular, assertions (ii) improves the corresponding result in [22, Corollary 6.21], which only asserts the existence of such convergence radius under the stronger assumption than that for assertions (ii). Recall that x^* is a local Pareto optimum of F and ξ^* is defined by (3.28).

THEOREM 4.2. *Suppose that F satisfies assumption* (3.3) *associated to* $(x^*; a^*, \frac{1}{a^*L})$ *with* $L(\cdot) \equiv L$. *Let* $x_0 \in \mathbf{B}(x^*, \frac{1}{2(1+a^*\xi^*)a^*L})$. *Then, with initial point x_0, we have the following assertions:*

(i) *The sequence $\{x_n\}$ generated by Algorithm 3.1 is well-defined and converges quadratically to a local Pareto optimum of F.*

(ii) *If* $\|x_0 - x^*\| \le \frac{-9(1-\sigma)^2+3(1-\sigma)\sqrt{1+9(1-\sigma)^2}}{(1+4a^*\xi^*)a^*L}$, *then any sequence $\{x_n\}$ generated by Algorithm 3.2 with initial point x_0 is well-defined and converges quadratically to a local Pareto optimum of F.*

Proof. Assertion (i) follows directly from Theorem 3.7. Then, it remains to verify assertion (ii). To do this, write $r := \frac{-9(1-\sigma)^2+3(1-\sigma)\sqrt{1+9(1-\sigma)^2}}{(1+4a^*\xi^*)a^*L}$. Then $r < \frac{1}{2(1+a^*\xi^*)a^*L}$ (due to the fact $-9(1-\sigma)^2 + 3(1-\sigma)\sqrt{1+9(1-\sigma)^2} < \frac{1}{2}$), and, $La^*\tau < \frac{1}{2(1+a^*\xi^*)} < \frac{1}{2}$ for each $\tau \in (0, r)$. As $L(\cdot) \equiv L$, one checks that, for each $\tau \in (0, r)$,

$$\beta_\tau = \frac{a^* \int_0^\tau L(u)(\tau-u)\mathrm{d}u + a^*\xi^*\tau}{1 - a^*\int_0^\tau L(u)\mathrm{d}u} = \frac{\frac{L}{2}a^*\tau^2 + a^*\xi^*\tau}{1 - La^*\tau} < (\frac{1}{2} + 2a^*\xi^*)\tau \le (\frac{1}{2} + 2a^*\xi^*)r.$$

Moreover, since $a_\tau L = \frac{a^*L}{1-a^*L\tau} < 2a^*L$, it follows that, for each $\tau \in (0, r)$,

$$\begin{aligned} \frac{3(1-\sigma)\left(1-a_\tau\int_0^{\bar{r}^*_{a_\tau}} L(\tau+u)\mathrm{d}u\right)}{a_\tau L(\tau+\bar{r}^*_{a_\tau})} &= \frac{3(1-\sigma)\sqrt{1-2a_\tau L\beta_\tau}}{a_\tau L} \\ &\ge \frac{3(1-\sigma)\sqrt{1-2(1+4a^*\xi^*)a^*Lr}}{2a^*L} \\ &= (\tfrac{1}{2} + 2a^*\xi^*)r, \end{aligned}$$

where the last equality holds by the definition of r. Thus, one checks that $r \in (0, \frac{b_{a^*}}{1+a^*\xi^*})$ satisfies (3.41), and the conclusion follows from Theorem 3.9. □

4.1.2. Global convergence of Algorithm 3.2. This subsection aims to establish the global convergence of Algorithm 3.2 under the classical Lipschitz condition.

The following proposition shows that any accumulation point of a sequence $\{x_n\}$ generated by Algorithm 3.2, where the stepsize $\{\alpha_n\}$ satisfies the Armijo rule, or the Goldstein rule, or the Wolfe rule, is a critical point of F.

PROPOSITION 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, any accumulation point x^* of $\{x_n\}$ such that $\mathrm{D}^2F(x^*)$ is positive definite and D^2F is Lipschitz continuous around x^*, is a local Pareto optimum of F.*

Proof. Let x^* be an accumulation point of $\{x_n\}$ such that $\mathrm{D}^2F(x^*)$ is positive definite and D^2F is Lipschitz continuous around x^*. Then, it is easy to show that $\mathrm{D}^2F(\cdot)$ is positive definite around x^*. By (2.2), we only need to verify that x^* is a critical point of F. As x^* is an accumulation point of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ such that $\lim_{i\to\infty} x_{n_i} = x^*$. Let $j \in \{1,\ldots,m\}$. Noting that $\{F_j(x_n)\}$ is monotonically nonincreasing (by Algorithm 3.2) and F_j is continuous, it follows that

$$\lim_{n\to\infty} F_j(x_n) = \lim_{i\to\infty} F_j(x_{n_i}) = F_j(x^*). \tag{4.9}$$

By (i) and (ii) of Lemma 2.4, to complete the proof, it suffices to verify that $\theta(x^*) \geq 0$. To do this, let

$$K_1 := \{i : F_j(x_{n_i} + s(x_{n_i})) \leq F_j(x_{n_i}) + \sigma\theta(x_{n_i}) \text{ for all } j = 1,\ldots,m\}.$$

Then, we divide the proof into two cases.

Case 1. K_1 is infinite. Then, there exists a subsequence of $\{x_{n_i}\}$, denoted by itself, such that

$$F_j(x_{n_i} + s(x_{n_i})) \leq F_j(x_{n_i}) + \sigma\theta(x_{n_i}) \quad \text{for all } i \in \mathbb{N} \text{ and } j = 1,\ldots,m. \tag{4.10}$$

In view of Step 4 of Algorithm 3.2, one has that $x_{n_i+1} = x_{n_i} + s(x_{n_i})$. Passing to the limit as $i \to \infty$ in (4.10), we get from (4.9) that $\theta(x^*) \geq 0$ and the proof is complete in this case.

Case 2. K_1 is finite. Then, there exist $j_0 \in \{1,\ldots,m\}$ and a subsequence of $\{x_{n_i}\}$, denoted by itself, such that

$$F_{j_0}(x_{n_i} + s(x_{n_i})) > F_{j_0}(x_{n_i}) + \sigma\theta(x_{n_i}) \quad \text{for all } i \in \mathbb{N}.$$

Thus, in view of Step 5 in Algorithm 3.2 (cf. (3.1)) and Lemma 2.4(i), we have

$$F_{j_0}(x_{n_i}) - F_{j_0}(x_{n_i+1}) \geq -\sigma\alpha_{n_i}\theta(x_{n_i}) \geq 0,$$

where each $\alpha_{n_i} \in (0,+\infty)$ satisfies the Armijo rule, or the Goldstein rule, or the Wolfe rule. This, together with (4.9), implies that $\lim_{i\to\infty} \alpha_{n_i}\theta(x_{n_i}) = 0$. Recall that θ is continuous around x^* (due to Lemma 2.4) and that $\lim_{i\to\infty} x_{n_i} = x^*$. We only need to consider the case when $\lim_{i\to\infty} \alpha_{n_i} = 0$ because, otherwise, one has that $\overline{\lim}_{i\to\infty} \alpha_{n_i} > 0$ and thus

$$\theta(x^*)\, \overline{\lim_{i\to\infty}}\, \alpha_{n_i} \geq \lim_{i\to\infty} \alpha_{n_i}\theta(x_{n_i}) = 0;$$

this implies $\theta(x^*) \geq 0$. To proceed, let $\zeta := \min\{\sigma, \nu\}$, and define for each n_i

$$\Theta(x_{n_i}) := \max_{k=1,2} \left\{ \frac{F_{j_0}(x_{n_i} + k\alpha_{n_i}s(x_{n_i})) - F_{j_0}(x_{n_i})}{k\alpha_{n_i}}, \nabla F_{j_0}(x_{n_i} + \alpha_{n_i}s(x_{n_i}))^T s(x_{n_i}) \right\}. \tag{4.11}$$

Then, $\zeta \in (0,1)$. Below, we show that

$$\overline{\lim_{i\to\infty}}\, \Theta(x_{n_i}) \le \theta(x^*) \quad \text{and} \quad \zeta\theta(x_{n_i}) \le \Theta(x_{n_i}) \;\text{ for each } n_i. \tag{4.12}$$

Granting this and noting $\lim_{i\to\infty} \theta(x_{n_i}) = \theta(x^*)$, one checks that $\theta(x^*) \ge \zeta\theta(x^*)$ and so $\theta(x^*) \ge 0$ (as $\zeta \in (0,1)$), completing the proof.

Note by Definition 3.1 that if α_{n_i} satisfies the Armijo rule, then

$$\Theta(x_{n_i}) \ge \frac{F_{j_0}(x_{n_i} + 2\alpha_{n_i} s(x_{n_i})) - F_{j_0}(x_{n_i})}{2\alpha_{n_i}} > \sigma\theta(x_{n_i}) \ge \zeta\theta(x_{n_i})$$

where the first and the last inequality holds by the definition of $\Theta(x_{n_i})$ (see (4.11)) and ζ, respectively. Similar argument is also valid for the Goldstein rule or the Wolfe rule, and thus the second relation in (4.12) is seen to hold. To show the first one in (4.12), we first note θ is continuous around x^* and $\{s(x_{n_i})\}$ is bounded (due to Lemma 2.4(iii)). Note further that ∇F_{j_0} is continuous. It follows from $\lim_{i\to\infty} \alpha_{n_i} = 0$ and the inequality $\nabla F_{j_0}(x_{n_i})^T s(x_{n_i}) \le \theta(x_{n_i})$ (due to the definition of θ) that

$$\begin{aligned}
&\overline{\lim}_{i\to\infty} \nabla F_{j_0}(x_{n_i} + \alpha_{n_i} s(x_{n_i}))^T s(x_{n_i}) \\
&\le \overline{\lim}_{i\to\infty}((\nabla F_{j_0}(x_{n_i} + \alpha_{n_i} s(x_{n_i})) - \nabla F_{j_0}(x_{n_i}))^T s(x_{n_i}) + \theta(x_{n_i})) \\
&= \overline{\lim}_{i\to\infty} \theta(x_{n_i}) = \theta(x^*).
\end{aligned}$$

Thus it remains to verify that

$$\overline{\lim_{i\to\infty}} \frac{F_{j_0}(x_{n_i} + k\alpha_{n_i} s(x_{n_i})) - F_{j_0}(x_{n_i})}{k\alpha_{n_i}} \le \theta(x^*) \quad \text{for } k = 1, 2. \tag{4.13}$$

To do this, consider a sequence $\{t_{n_i}\} \subseteq (0, +\infty)$ converging to zero. Then we have that

$$\lim_{i\to\infty} \int_0^1 (\nabla F_{j_0}(x_{n_i} + \tau t_{n_i} s(x_{n_i})) - \nabla F_{j_0}(x_{n_i}))^T s(x_{n_i}) \mathrm{d}\tau = 0 \tag{4.14}$$

as ∇F_{j_0} is continuous and $\{s(x_{n_i})\}$ is bounded. Note for each $i \in \mathbb{N}$ that

$$\begin{aligned}
\frac{F_{j_0}(x_{n_i} + t_{n_i} s(x_{n_i})) - F_{j_0}(x_{n_i})}{t_{n_i}} \;=\; & \int_0^1 (\nabla F_{j_0}(x_{n_i} + \tau t_{n_i} s(x_{n_i})) - \nabla F_{j_0}(x_{n_i}))^T s(x_{n_i}) \mathrm{d}\tau \\
& + \nabla F_{j_0}(x_{n_i})^T s(x_{n_i}).
\end{aligned}$$

Hence, thanks again to the inequality $\nabla F_{j_0}(x_{n_i})^T s(x_{n_i}) \le \theta(x_{n_i})$ (due to the definition of θ) and using again the continuity of θ, we conclude from (4.14) that

$$\overline{\lim_{i\to\infty}} \frac{F_{j_0}(x_{n_i} + t_{n_i} s(x_{n_i})) - F_{j_0}(x_{n_i})}{t_{n_i}} \le \overline{\lim_{i\to\infty}}\, \theta(x_{n_i}) = \theta(x^*).$$

Applying this fact to $\{\alpha_{n_i}\}$ and $\{2\alpha_{n_i}\}$ in place of $\{t_{n_i}\}$, one sees that (4.13) holds, and the proof is complete. □

COROLLARY 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Suppose that the set $\bigcap_{j=1,\dots,m}\{x \in U : F_j(x) \le F_j(x_0)\}$ is bounded. Then, there exists an accumulation point*

x^* of $\{x_n\}$. Furthermore, if x^* satisfies that $\mathrm{D}^2F(x^*)$ is positive definite and D^2F is Lipschitz continuous around x^*, then x^* is a local Pareto optimum of F.

Proof. Note by Algorithm 3.2 that $\{F_j(x_n)\}$ is monotonically nonincreasing for each $j = 1, \ldots, m$. Hence, by assumption, we have that $\{x_n\} \subseteq \bigcap_{j=1,\ldots,m}\{x \in U : F_j(x) \le F_j(x_0)\}$ and so $\{x_n\}$ is bounded. Thus, there exists an accumulation point of $\{x_n\}$. Then, the conclusion follows from Proposition 4.3. □

Now we are ready to establish the global quadratic convergence of a sequence generated by Algorithm 3.2.

THEOREM 4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Suppose that $\{x_n\}$ has an accumulation point x^* such that $\mathrm{D}^2F(x^*)$ is positive definite and D^2F is Lipschitz continuous around x^*. Then, x^* is a local Pareto optimum of F and $\{x_n\}$ converges quadratically to x^*.*

Proof. In view of Proposition 4.3, it suffices to show that $\{x_n\}$ converges quadratically to x^*. For this purpose, note by the Lipschitz continuity assumption that there exists a pair of positive numbers (r, L) such that each D^2F satisfies the Lipschitz condition with modulus L on $\mathbf{B}(x^*, r)$. Since each $\nabla^2 F_j(x^*)$ is positive definite by assumption, we can take

$$a^* > \max_{j=1,\ldots,m}\left\{\frac{1}{rL}, \|\nabla^2 F_j(x^*)^{-1}\|\right\}.$$

Then, F satisfies assumption (3.3) associated to $(x^*; a^*, \frac{1}{a^*L})$ and $L(\cdot) \equiv L$. Let

$$\hat{r} = \frac{-9(1-\sigma)^2 + 3(1-\sigma)\sqrt{1+9(1-\sigma)^2}}{(1+4a^*\xi^*)a^*L},$$

and let $\{x_{n_i}\} \subseteq \{x_n\}$ be a subsequence such that $\lim_{i\to\infty} x_{n_i} = x^*$. Then there exists $i_0 \in \mathbb{N}$ such that $\|x_{n_{i_0}} - x^*\| \le \hat{r}$. Thus, Theorem 4.2(ii) is applicable to concluding that the sequence $\{x_n\}_{n=n_{i_0}}^{\infty}$ converges quadratically to a local Pareto optimum of F. This completes the proof. □

4.2. Theorems under the γ-condition. The notion of the γ-condition was introduced by Wang in [42] for differentiable operator, and was used to improve Smale's corresponding results for convergence analysis of the Newton method (cf. [40]). Below, we present an analogue of the γ-condition (with a slight modification). Let $r > 0$ and $\gamma > 0$ be such that $r\gamma \le 1$.

DEFINITION 4.6. *Let $x_0 \in U$ and $r > 0$ be such that $\mathbf{B}(x_0, r) \subseteq U$. $\mathrm{D}F$ is said to satisfy the γ-condition on $\mathbf{B}(x_0, r)$ if*

$$\|\nabla^3 F_i(x)\| \le \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3} \quad \text{for each } i \in \{1, \ldots, m\} \text{ and } x \in \mathbf{B}(x_0, r).$$

REMARK 4.2. *As in [42], one checks by definition that if F is analytic at x_0, then $\mathrm{D}F$ satisfies the γ-condition on $\mathbf{B}(x_0, \frac{1}{\gamma})$, where $\gamma := \max\limits_{i=1,\ldots,m}\{\sup_{k\ge 2}\|\frac{1}{k!}F_i^{(k+1)}(x_0)\|^{\frac{1}{k-1}}\}$.*

The following proposition shows that the γ-condition of $\mathrm{D}F$ implies the L-average Lipschitz condition of D^2F, the proof of which is easy and so is omitted here.

Proposition 4.7. *Suppose that* DF *satisfies the* γ*-condition on* $\mathbf{B}(x_0, r)$. *Then,* D^2F *satisfies the* L*-average Lipschitz condition on* $\mathbf{B}(x_0, \frac{1}{\gamma})$ *with the function* $L : [0, \frac{1}{\gamma}) \to \mathbb{R}_+$ *defined by*

$$L(u) := \frac{2\gamma}{(1-\gamma u)^3} \quad \textit{for each } u \in [0, \frac{1}{\gamma}). \tag{4.15}$$

Let $a > 0$ and $\beta \geq 0$. For $L(\cdot)$ given by (4.15), the majoring function h_a defined in (2.13) is reduced to

$$h_a(t) = \beta - t + \frac{a\gamma t^2}{1-\gamma t} \quad \text{for each } 0 \leq t < \frac{1}{\gamma}.$$

Then, it follows from (2.12) that

$$r_a = \left(1 - \sqrt{\frac{a}{1+a}}\right)\frac{1}{\gamma} \quad \text{and} \quad b_a = \left(1 + 2a - 2\sqrt{a(1+a)}\right)\frac{1}{\gamma}.$$

Let $\{t_{a,n}\}$ denote a sequence generated by the classical Newton method for approaching the zeros of h_a with the initial value $t_0 = 0$, and assume

$$\gamma\beta \leq 1 + 2a - 2\sqrt{a(1+a)}.$$

Then, by [42, p.180], the smaller zero r_a^* of h_a and the Newton sequence $\{t_{a,n}\}$ have the following closed forms:

$$r_a^* = \frac{1+\gamma\beta-\sqrt{\varrho}}{2(1+a)\gamma}, \quad \text{and} \quad t_{a,n} = \frac{1-\mu^{2^n-1}}{1-\mu^{2^n-1}\eta} r_a^* \quad \text{for each } n \in \mathbb{N}, \tag{4.16}$$

where $\varrho := (1+\gamma\beta)^2 - 4(1+a)\gamma\beta \geq 0$,

$$\mu := \frac{1-\gamma\beta-\sqrt{\varrho}}{1-\gamma\beta+\sqrt{\varrho}} \quad \text{and} \quad \eta := \frac{1+\gamma\beta-\sqrt{\varrho}}{1+\gamma\beta+\sqrt{\varrho}}. \tag{4.17}$$

Fixing the triple $(x; a, r)$ with $x \in U$ and $(a, r) \in \mathbb{R}_+^2$, we consider the following assumption for $F \in C^3(U, \mathbb{R}^m)$ associated to the triple $(x; a, r)$:

- D$^2F(x)$ is positive definite with each $\|\nabla^2 F_i(x)^{-1}\| \leq a$;
- DF satisfies the γ-condition on $\mathbf{B}(x, r) \subseteq U$. (4.18)

Then, we have the following theorem about the quadratic convergence criterion of the extended Newton method under the γ-condition.

Theorem 4.8. *Suppose that* $\|s(x_0)\| \leq \beta$ *and* F *satisfies assumption* (4.18) *associated to* $(x_0; a, r_a^*)$. *Let* μ *and* η *be given by* (4.17). *Then, with initial point* x_0, *we have the following assertions:*

(i) *If* $\beta \leq \left(1 + 2a - 2\sqrt{a(1+a)}\right)\frac{1}{\gamma}$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 is well-defined, stays in* $\mathbf{B}(x_0, r_a^*)$, *and converges to a local Pareto optimum* $\bar{x} \in \mathbf{B}[x_0, r_a^*]$ *with the following error estimate for each* $n \in \mathbb{N}$:

$$\|x_n - \bar{x}\| \leq \frac{(1-\eta)\mu^{2^n-1}}{1-\mu^{2^n-1}\eta} r_a^*. \tag{4.19}$$

(ii) *If* $\beta < \left(1+2a-2\sqrt{a(1+a)}\right)\frac{1}{\gamma}$, *then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges quadratically to* $\bar{x}$ *with the following error estimate for some* $N \in \mathbb{N}$*:*

$$\|x_{n+1}-\bar{x}\| \leq \frac{\mu(1-\mu^{2^{n+1}-1}\eta)(1-\mu^{2^n-1}\eta)^2}{(1-\eta)(1-\mu^{2^n}(2-\mu^{2^n-1}\eta))^2 r_a^*}\|x_n-\bar{x}\|^2 \quad \text{for each } n \geq N. \tag{4.20}$$

(iii) *If* $\beta \leq \frac{3(1-\sigma)(1-\gamma\beta)(1-2\gamma\beta(1+2a)+\gamma^2\beta^2)}{2a\gamma(1+\gamma\beta)^3}$, *then any sequence* $\{x_n\}$ *generated by Algorithm 3.2 coincides with the one generated by Algorithm 3.1, and satisfies* (4.19) *and* (4.20)*.*

Proof. With L defined by (4.15), one checks that $\gamma\int_0^{\frac{1}{\gamma}} L(u)\left(\frac{1}{\gamma}-u\right)\mathrm{d}u = +\infty$ and so (2.11) holds with $\frac{1}{\gamma}$ in place of R. This and assumption (4.18) in combination with Proposition 4.7 imply that F satisfies assumption (3.3) associated to $(x_0; a, r_a^*)$ and L. Hence, Theorem 3.4 is applicable to concluding that assertions (i) and (ii) hold. Then, it remains to show assertion (iii). In fact, as $L(\cdot)$ is given by (4.15), it follows that

$$\frac{3(1-\sigma)\left(1-a\int_0^{r_a^*} L(u)\mathrm{d}u\right)}{aL(r_a^*)} = \frac{3(1-\sigma)(1-r_a^*\gamma)((1+a)(1-r_a^*\gamma)^2-a)}{2a\gamma}. \tag{4.21}$$

Note further by (4.16) that

$$r_a^*\gamma = \frac{1+\gamma\beta-\sqrt{\varrho}}{2(1+a)} = \frac{(1+\gamma\beta)^2-\varrho}{2(1+a)(1+\gamma\beta+\sqrt{\varrho})} \leq \frac{2\gamma\beta}{1+\gamma\beta}.$$

Combing this with (4.21) gives that

$$\frac{3(1-\sigma)(1-\gamma\beta)(1-2\gamma\beta(1+2a)+\gamma^2\beta^2)}{2a\gamma(1+\gamma\beta)^3} \leq \frac{3(1-\sigma)\left(1-a\int_0^{r_a^*} L(u)\mathrm{d}u\right)}{aL(r_a^*)}.$$

Thus, if $\beta \leq \frac{3(1-\sigma)(1-\gamma\beta)(1-2\gamma\beta(1+2a)+\gamma^2\beta^2)}{2a\gamma(1+\gamma\beta)^3}$, then (3.19) holds. Hence, the conclusion follows from Proposition 4.7 and Theorem 3.5. □

Similarly, we have the following results by using Theorem 3.7 in combination with Proposition 4.7, regarding an estimate of the radius of the convergence ball of the extended Newton method for multiobjective optimization under the γ-condition. Recall that x^* is a local Pareto optimum of F and ξ^* is defined by (3.28).

THEOREM 4.9. *Suppose that* F *satisfies assumption* (4.18) *associated to* $(x^*; a^*, r_{a^*})$*. Let* $x_0 \in \mathbf{B}\left(x^*, \frac{1+2a^*-2\sqrt{a^*(1+a^*)}}{(1+a^*\xi^*)\gamma}\right)$*. Then, with initial point* x_0*, we have the following assertions:*

(i) *The sequence* $\{x_n\}$ *generated by Algorithm 3.1 is well-defined and converges quadratically to a local Pareto optimum of* F*.*

(ii) *Let* $0 < r < \frac{1+2a^*-2\sqrt{a^*(1+a^*)}}{(1+a^*\xi^*)\gamma}$ *satisfy* (3.41)*. Then for any* $x_0 \in \mathbf{B}(x^*, r)$*, any sequence* $\{x_n\}$ *generated by Algorithms 3.2 with initial point* x_0 *converges quadratically to a local Pareto optimum of* F*.*

The advantage of considering the L-average Lipschitz condition rather than the classical Lipschitz condition is shown in the following example, for which Theorem 4.8 is applicable but not Theorem 4.1.

EXAMPLE 4.2. Consider problem (1.1) with $m = l = 1$ and $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \begin{cases} (\tau - 1)x - \ln(1 - x), & x \le \frac{1}{2}, \\ (\tau + 1)x - 2x^2 + \frac{8}{3}x^3 - \frac{5}{6} + \ln 2, & x \ge \frac{1}{2}. \end{cases}$$

where $\tau \in (10\sqrt{2} - 14, 3 - 2\sqrt{2})$. Then one checks that

$$F''(x) = \begin{cases} \frac{1}{(1-x)^2}, & x \le \frac{1}{2}, \\ -4 + 16x, & x \ge \frac{1}{2}, \end{cases} \quad \text{and} \quad F^{'''}(x) = \begin{cases} \frac{2}{(1-x)^3}, & x \le \frac{1}{2}, \\ 16, & x \ge \frac{1}{2}. \end{cases}$$

Let $x_0 := 0$ and $\gamma := 1$. It follows that $a := \|F''(x_0)^{-1}\| = 1$, and that F' satisfies the γ-condition on $\mathbf{B}(x_0, 1)$. Note that

$$\beta := \|s(x_0)\| = \| - (F''(x_0))^{-1}F'(x_0)\| = \tau < 3 - 2\sqrt{2}.$$

Therefore, Theorem 4.8 is applicable to concluding that the sequence $\{x_n\}$ generated by Algorithm 3.1 with initial point x_0 converges to a local Pareto optimum of F. We below show that Theorem 4.1 is not applicable. To do this, we first note that F'' is also Lipschitz continuous on $\mathbf{B}(x_0, r)$ with the smallest Lipschitz constant K_r given by

$$K_r := \begin{cases} \frac{2}{(1-r)^3}, & r \le \frac{1}{2}, \\ 16, & r \ge \frac{1}{2}. \end{cases} \tag{4.22}$$

Now suppose on the contrary that Theorem 4.1 is applicable. Then there exists a positive constant L such that

$$L \ge K_r, \quad r \ge \frac{1 - \sqrt{1 - 2L\tau}}{L} \quad \text{and} \quad \tau \le \frac{1}{2L} \le \frac{1}{2K_r}, \tag{4.23}$$

as $a = 1$ and $\beta = \tau$. Recalling $\tau > 10\sqrt{2} - 14 > \frac{1}{32}$, we have that $K_r < 16$, and then it follows from (4.22) that $r < \frac{1}{2}$. Hence $L \ge K_r = \frac{2}{(1-r)^3} \ge 2$. Consequently, by the second inequality in (4.23), we have that $\tau \le r - \frac{Lr^2}{2}$ and so $\tau \le r - r^2$. Combining this and the last inequaliity in (4.23), and (4.22), we have that $\tau \le \min\{\frac{(1-r)^3}{4}, r - r^2\}$. Since the function $r \mapsto \frac{(1-r)^3}{4}$ is decreasing and $r \mapsto r - r^2$ increasing on $[0, \frac{1}{2}]$, it follows that, for each $r \in (0, \frac{1}{2})$,

$$\min\left\{\frac{(1 - r)^3}{4}, r - r^2\right\} \le s_0 - s_0^2 = 10\sqrt{2} - 14,$$

where $s_0 := 3 - 2\sqrt{2}$ is the least positive root of equation $\frac{(1-s)^3}{4} = s - s^2$. Therefore, $\tau \le 10\sqrt{2} - 14$, which contradicts the choice of τ, and thus Theorem 4.1 is not applicable.

We end this section with the following example, which concerns nontrivial examples of functions (F, L) satisfying the L-average Lipschitz condition.

EXAMPLE 4.3. Let $\{\gamma_n\}$ be a positive sequence, and let $F : U \subseteq \mathbb{R}^l \to \mathbb{R}^m$ be an analytic operator (at least locally around the point x_0 under consideration). Let $x_0 \in U$. Suppose that

$$\max_{j=1,\cdots,m} \|F_j^{(n+2)}(x_0)\| \le \gamma_n \quad \text{for each } n \ge 1. \tag{4.24}$$

Let $\gamma, c \in (0, +\infty)$ and $p \in (-1, 0) \cup (0, +\infty)$. Below, we consider some special examples of the sequence $\{\gamma_n\}$ used by Wang in [42]:

$$\begin{array}{ll} \text{Exponential type:} & \{\gamma_n\} := \{c\gamma^n\}; \\ \text{Binomial type:} & \{\gamma_n\} := \{c\frac{(p+n)!}{p!}\gamma^n\}; \\ \text{The first logarithmic type:} & \{\gamma_n\} := \{cn!\gamma^n\}; \\ \text{The second logarithmic type:} & \{\gamma_n\} := \{c(n-1)!\gamma^n\}. \end{array} \tag{4.25}$$

Then, one can check, as done in [42], that D^2F satisfies the L-average Lipschitz condition on $\mathbf{B}(x_0, R)$ with L defined by

$$L(u) := g''(u) \quad \text{for each } 0 \le u < R, \tag{4.26}$$

and the majorizing function h_a is given by

$$h_a(t) := \beta - t + ag(t) \quad \text{for each } t \in [0, R),$$

where R and the function $g : [0, R) \to \mathbb{R}$ corresponding to the sequences $\{\gamma_n\}$ given by (4.25) are listed in Table 4.1 below.

TABLE 4.1
Values of R and g.

γ_n	R	$g(t)$
$c\gamma^n$	$+\infty$	$\frac{c}{\gamma}\left(e^{\gamma t} - \gamma t - 1\right)$
$c\frac{(p+n)!}{p!}\gamma^n$	$\frac{1}{\gamma}$	$\frac{c}{p\gamma}\Big((1-\gamma t)^{-p} - p\gamma t - 1\Big)$
$cn!\gamma^n$	$\frac{1}{\gamma}$	$\frac{c}{\gamma}\ln\frac{1}{1-\gamma t} - ct$
$c(n-1)!\gamma^n$	$\frac{1}{\gamma}$	$\frac{c}{\gamma}(1-\gamma t)\ln(1-\gamma t) + ct$

Let $a > 0$ and assume further that $a > \frac{1}{c}$ in the case when $\{\gamma_n\}$ is the second logarithmic type. Then (2.11) holds because $\frac{1}{R}\int_0^R L(u)(R-u)\mathrm{d}u = \lim_{u\to R^-} \frac{1}{u}g(u)$ is equal to c if $\{\gamma_n\}$ is the second logarithmic type, and to $+\infty$ otherwise. Thus, assuming (4.24) with $\{\gamma_n\}$ given by each in (4.25), F satisfies assumption (3.3) associated to $(x_0; a, r_a)$ (and so to $(x_0; a, r_a^*)$) and L defined by (4.26). Therefore, Theorems 3.4-3.5, and Theorems 3.7-3.9 (assuming (4.24) with x^* in place of x_0, and $a^* > \frac{1}{c}$ in the case when $\{\gamma_n\}$ is the second logarithmic type) are applicable to establish the corresponding results regarding the convergence criteria and the radius of convergence balls for Algorithms 3.1 and 3.2, respectively. As illustrating examples, we provide in the following Table 4.2 the values of b_a and the radius $r = \frac{b_{a^*}}{1+a^*\xi^*}$ of the convergence balls for Algorithm 3.1, where ξ^* is defined by (3.28).

TABLE 4.2
Values of b_a and r.

γ_n	b_a	r
$c\gamma^n$	$\frac{ac+1}{\gamma}\ln\frac{ac+1}{ac}-\frac{1}{\gamma}$	$\frac{1}{\gamma(1+a^*\xi^*)}\left(a^*c+1)\ln\frac{a^*c+1}{a^*c}-1\right)$
$c\frac{(p+n)!}{p!}\gamma^n$	$\frac{1}{\gamma}+\frac{ac}{\gamma}\frac{p+1}{p}\left(1-\left(\frac{ac+1}{ac}\right)^{\frac{p}{p+1}}\right)$	$\frac{1}{\gamma(1+a^*\xi^*)}\left(1+a^*c\frac{p+1}{p}\left(1-\left(\frac{a^*c+1}{a^*c}\right)^{\frac{p}{p+1}}\right)\right)$
$cn!\gamma^n$	$\frac{1}{\gamma}-\frac{ac}{\gamma}\ln\frac{ac+1}{ac}$	$\frac{1}{\gamma(1+a^*\xi^*)}\left(1-a^*c\ln\frac{a^*c+1}{a^*c}\right)$
$c(n-1)!\gamma^n$	$\frac{1}{\gamma}-\frac{ac}{\gamma}+\frac{ac}{\gamma}e^{-\frac{1}{ac}}$	$\frac{1}{\gamma(1+a^*\xi^*)}\left(1-a^*c+a^*ce^{-\frac{1}{a^*c}}\right)$

5. Numerical experiments. The purpose of this section is to carry out some numerical experiments and demonstrate the numerical performance of the extended Newton method for some multiobjective optimization problems. All numerical experiments are implemented in Matlab R2014a and executed on a personal desktop (Intel Core Duo i7-6700, 3.40 GHz, 8.00 GB of RAM).

Two classical bi-objective optimization problems are tested as follows. Example 5.1 is taken from [25] and has been tested as a benchmark problem in various works; see [22, 25] and references therein. Example 5.2 is an extension of Example 5.1 to the negative likelihood function of logistic regression.

EXAMPLE 5.1. Consider problem (1.1) with $F:\mathbb{R}^n\to\mathbb{R}^2$ defined by

$$F_1(x):=\frac{1}{n}\|x\|^2 \quad \text{and} \quad F_2(x):=\frac{1}{n}\|x-2e\|^2 \quad \text{for each } x\in\mathbb{R}^n,$$

where e denotes the vector of ones in $\mathbb{R}^n$.

EXAMPLE 5.2. Consider problem (1.1) with $F:\mathbb{R}^n\to\mathbb{R}^2$ defined by

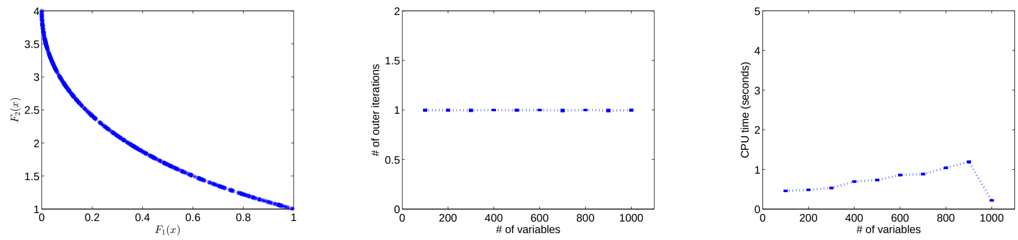
$$F_1(x):=\frac{1}{n}\|x\|^2 \quad \text{and} \quad F_2(x):=-\frac{1}{m}\sum_{i=1}^{m}\log(1+\exp(-b_ix^Ta_i)) \quad \text{for each } x\in\mathbb{R}^n,$$

where $A:=(a_1,\ldots,a_m)\in\mathbb{R}^{n\times m}$ and $b\in\mathbb{R}^m$ are randomly generated i.i.d. Gaussian ensembles according to the logistic regression model (cf. [10]).

For each test problem, we set the dimension of variables n from [100,1000], and for each n, we apply the extended Newton method to solve the corresponding problem in 500 simulations by using random initial points from a joint uniform random distribution. In particular, the initial points in Example 5.1 are randomly selected via the Matlab script $x_0:=2{*}rand{*}rand(n,1)$, and the ones in Example 5.2 are $x_0:=-\sqrt{n}{*}rand{*}rand(n,1)$. The solver and the parameters used in the extended Newton method are described as follows. The subproblem (2.3) of finding Newton direction is implemented by adopting the CVX solver to solve the corresponding problem (3.2); see Remark 3.1 for the explanation. We use the Armijo line-search with $\sigma=0.1$ and set the stopping criterion of the extended Newton method as $\theta(x_k)\le$ 1e-6 or the number of iterations is greater than 100.

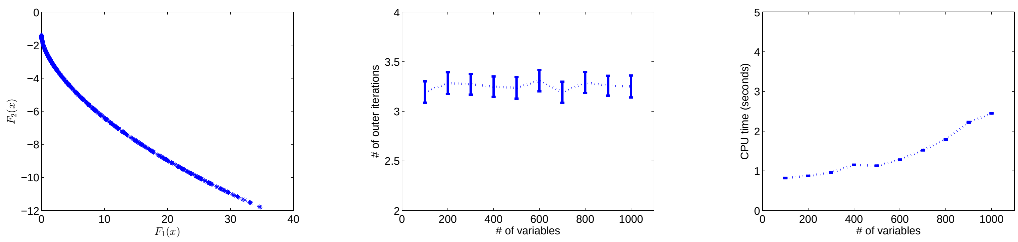
By taking the average of these 500 simulations, the numerical results of applying the extended Newton method to solve Examples 5.1 and 5.2 are illustrated in Figures 5.1 and 5.2, respectively. In these figures, subfigures (a) plot the Pareto frontier generated by the extended Newton method when $n = 100$ and by using different (random) initial points, subfigures (b) plot the error bars of the number of outer iterations used by the extended Newton method in these 500 simulations along with the dimensions of variables, and subfigures (c) plot the error bars of the cost CPU time (in seconds) for solving each subproblem (2.3) in these 500 simulations along with the dimensions of variables.

Three observations are indicated from Figures 5.1 and 5.2 consistently: (a) most of the Pareto frontier could be constructed by the extended Newton method via using many different (random) initial points; (b) the extended Newton method usually converges very fast, and particularly, achieves a Pareto solution within only a few iterations; (c) the subproblem (2.3) could be solved efficiently by the CVX solver, even for large-scale problems.



(a) Pareto frontier ($n = 100$) (b) Number of outer iterations (c) CPU time

FIG. 5.1. *Numerical performance of the extended Newton method in Example 5.1.*



(a) Pareto frontier ($n = 100$) (b) Number of outer iterations (c) CPU time

FIG. 5.2. *Numerical performance of the extended Newton method in Example 5.2.*

Finally, we test on two more challenging bi-objective optimization problems. Example 5.3 is a convex problem taken from [22], and Example 5.4 is a 2-dimensional nonconvex problem taken from [36].

EXAMPLE 5.3. Consider problem (1.1) with $F : \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$F_1(x) := \frac{1}{n^2}\sum_{i=1}^{n} i(x_i - i)^4 \quad \text{and} \quad F_2(x) := \frac{1}{n(n+1)}\sum_{i=1}^{n} i(n-i+1)e^{-x_i} \quad \text{for each } x \in \mathbb{R}^n.$$

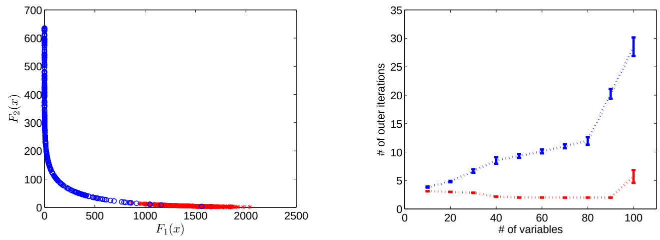
EXAMPLE 5.4. Consider problem (1.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_1(x) := x_1^4 + x_2^4 - x_1^2 + x_2^2 - 10x_1x_2 + 10x_1 + 20 \quad \text{and} \quad F_2(x) := \|x - e\|^2 \quad \text{for each } x \in \mathbb{R}^2.$$

The experiment setting and the implementation of the extended Newton method are similar to the preceding ones, except what is mentioned below. The dimension of variables are set from [10,100] in Example 5.3. The initial points in Example 5.3 are selected via two random strategies: (i) $x_0 := 2 * rand(n, 1)$ and (ii) $x_0 := 2 * rand(n, 1). * [1 : n]'$, and the ones in Example 5.4 are randomly selected from a normal distribution $x_0 := 2 * randn(n, 1)$. For the nonconvex Example 5.4, the Matlab solver "fminunc" is used to solve the corresponding nonconvex subproblem (2.3).

The numerical results are illustrated in Figures 5.3 and 5.4, respectively. In these figures, subfigures (a) plot the Pareto frontier generated by the extended Newton method when using different (random) initial points, and subfigures (b) plot the number of outer iterations used by the extended Newton method in the simulation trials.

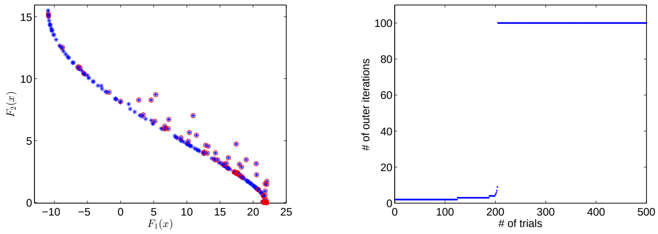
In Example 5.3, it is observed from Figure 5.3 that (a) most of the Pareto frontier could be constructed by the extended Newton method via using the two random initialization strategies; (b) the extended Newton method usually converges fast and stably, although it employs more iterations than the preceding experiments. Each subproblem can be solved by CVX in 1 second.



(a) Pareto frontier ($n = 10$) (b) Number of outer iterations

FIG. 5.3. *Numerical performance of the extended Newton method in Example 5.3. The blue and red symbols denote the different random initialization strategies,* $x_0 := 2 * rand(n, 1)$ *and* $x_0 := 2 * rand(n, 1). * [1 : n]'$*, respectively.*

In Example 5.4, it is indicated from Figure 5.4(a) that some of the Pareto frontier could be constructed by the extended Newton method, but some of the estimated solutions (over 25%) are not the Pareto optimum of this problem. It is demonstrated from Figure 5.4(b) that about 40% trials can be efficiently solved by the extended Newton method within a few iterations, while others cannot. The main reason of failure could be that the iterative sequence falls into some local optimum. In a word, the extended Newton method is an efficient numerical algorithm for convex bi-objective optimization problems, but may be not effective for the nonconvex problems.



(a) Pareto frontier (b) Number of outer iterations

Fig. 5.4. *Numerical performance of the extended Newton method in Example 5.4. The red circle in (a) denotes the obtained solution that is not a Pareto optimum.*

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